Nuclear Direct Reactions to Continuum 1

– How to get Nuclear Structure Information –

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I. Introduction

● What we want to discuss

Put a certain impulse on the nucleus
by **Nuclear Reaction**

$\Longrightarrow$

Observe its **Responses**

$\Longrightarrow$

Get information about **Nuclear Structure**

図 1 Hammer test of bridge

● What's promising method ?

- Clean impulse & Clean target

$\Longrightarrow$ Clean responses

$\Longrightarrow$ Clear information

- Dirty impulse & Dirty target

$\Longrightarrow$ Dirty responses

$\Longrightarrow$ Need a lot of efforts to get meaningful information

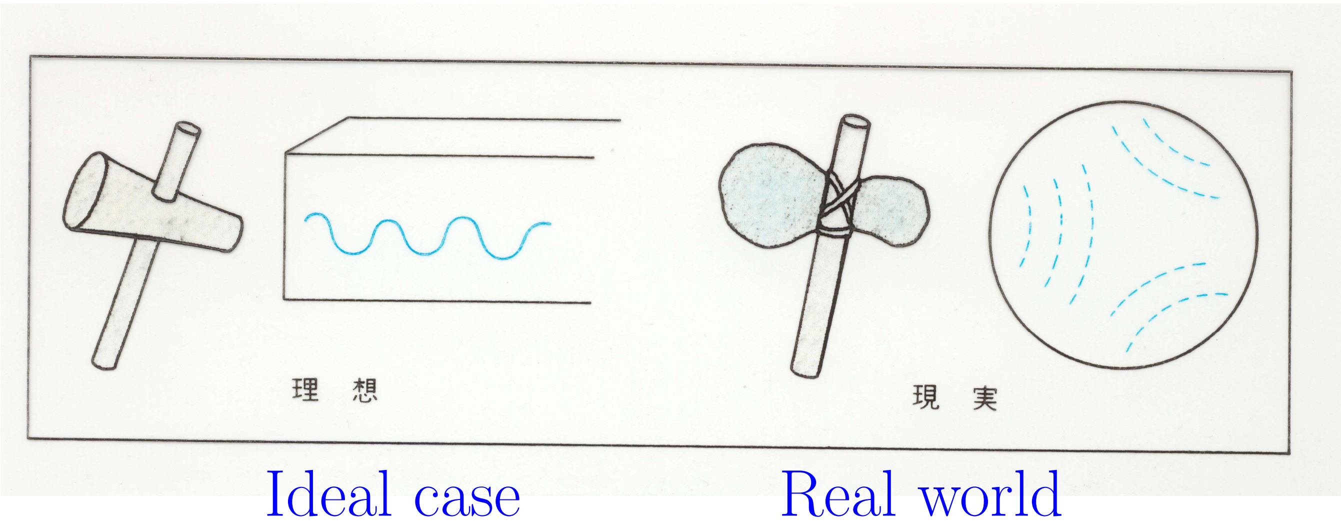


図 2 M. Ichimura, Parity vol. 03, no. 12 (1988) 10

- What reactions produce clean impulses ?
  - Simple reaction mechanisms are favored.
  - Direct Reaction is promising.

- **Direct Reaction**
  - Only a few degrees of freedom involve.
  - Short reaction time
    (No time for complicate processes)
  - Only a few step processes involve.

Consequently

- Higher energy reactions are favored.
- Weaker interactions are favored.
- Single step processes are most favorite.
- Simple probes are favored.

  Such as electron (e), nucleons (N), $\cdots$

In this lecture, I mostly restrict myself to **Single-step Reactions**, described by Distorted Wave Impulse Approximation (DWIA).

● Discrete vs. Continuum

For reactions to a discrete state,

$$a + A \longrightarrow b + B \text{ (isolated discrete state)}$$

(a and b are structureless particles),

the impulses may be well characterized by

Transferred energy ω
Transferred spin $J_{\rm tr}$
Transferred parity $\pi_{\rm tr}$
(Transferred isospin $T_{\rm tr}$)
etc.

Theories for these reactions can be found in the standard textbooks.

However, if the final states of B is in Continuum (unbound states),

Transferred quantum numbers are hard to be distinguished

Standard methods encounter serious problems in practical calculations

We need to develop proper methods to cope with them.

For this purpose, I will discuss Response Function method

To design experiments. it is important to choose proper impulses (reactions)

We must care of

- Choice of probes
- Selection rules
- Choice of energy
- Choice of angles

etc.

I hope this lecture helps a little.

- Examples

- Clean impulse

  Ex. (e, e'), (ν, ν')

  Use electric or weak interaction

- Dirty impulse

  Heavy ion reactions are usually very complicated. Need special trick !

- Relatively clean impulse

  High energy nucleon scatterings

  ex. $(p, p'), (p, n), (n, p)$

In this lecture, I mostly discuss nucleon induced reactions.

II. Basics of Reaction Theory

1. Convention

1.1 Natural unit

We use the natural unit in this lecture

$$\hbar = 1, \quad c = 1$$

Don't worry, just remember

$$\hbar c \approx 200 \text{ MeV} \cdot \text{fm}(= 197.326968)$$

[**Exercise 1**] Pion mass $m_\pi = 140$ MeV.
Calculate its Compton wave length in fm.

(Ans.)

$$\frac{\lambda}{2\pi} = \frac{1}{m_\pi} = \frac{\hbar c}{140\text{MeV}} = \frac{200 \text{ MeV} \cdot \text{fm}}{140\text{MeV}} = 1.4 \text{ fm}$$

1.2 Expression of the plane wave

In this lecture I express
the **Plane wave with momentum $\boldsymbol{p}$** as

$$\phi_{\boldsymbol{p}}(\boldsymbol{r}) = \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}$$

- Normalization

$$\langle \phi_{\boldsymbol{p}'} | \phi_{\boldsymbol{p}} \rangle = (2\pi)^3 \delta(\boldsymbol{p}' - \boldsymbol{p})$$

- Number of states in the phase volume $d^3\boldsymbol{p}$

$$dn = \frac{d^3\boldsymbol{p}}{(2\pi)^3}$$

Be careful about the convention.
Different textbooks use different conventions,
then the formulas of cross sections, etc.
are different correspondingly.

[Just for fun] (= Appendix)

Other expressions of the plane wave

(1) Momentum normalization

$$\phi_{\boldsymbol{p}}(\boldsymbol{r}) = \frac{1}{\sqrt{(2\pi)^3}} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}$$

$$\langle \phi_{\boldsymbol{p}'} | \phi_{\boldsymbol{p}} \rangle = \delta(\boldsymbol{p}' - \boldsymbol{p})$$

(2) Energy normalization (non-relativistic)

$$\phi_{\boldsymbol{p},E}(\boldsymbol{r}) = \sqrt{\frac{mp}{(2\pi)^3}} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}$$

with

$$E = \frac{p^2}{2m}$$

$$\langle \phi_{\boldsymbol{p}',E'} | \phi_{\boldsymbol{p},E} \rangle = \delta(E' - E)\delta(\Omega_{\boldsymbol{p}'} - \Omega_{\boldsymbol{p}})$$

[Comment]

- Box normalization

$$\phi_{\boldsymbol{p}_n}(\boldsymbol{r}) = \frac{1}{\sqrt{L^3}} \mathrm{e}^{\mathrm{i}\boldsymbol{p}_n \cdot \boldsymbol{r}}$$

$$\boldsymbol{p}_n = \left(\frac{2\pi}{L} n_x, \frac{2\pi}{L} n_y, \frac{2\pi}{L} n_z \right)$$

$$\langle \phi_{\boldsymbol{p}_{n'_x n'_y n'_z}} | \phi_{\boldsymbol{p}_{n_x n_y n_z}} \rangle = \delta_{n'_x n_x} \delta_{n'_y n_y} \delta_{n'_z n_z}$$

$$dn = dn_x dn_y dn_z = \frac{L^3}{(2\pi)^3} d^3\boldsymbol{p}$$

Set $L^3 = 1$, then get our convention

2. Relativistic Kinematics

2.1. One particle system

Energy-momentum vector (4-momentum)

$$p^\mu = (E, \boldsymbol{p})$$

On mass shell

$$p^\mu p_\mu = E^2 - \boldsymbol{p}^2 = m^2$$

Velocity

$$\boldsymbol{v} = \frac{\boldsymbol{p}}{E}$$

[**Exercise 2**] 300 MeV proton beam.
How much the velocity is ?

(Ans.) $m_p = 938$ MeV, $T_p = 300$ MeV

$$E_p = m_p + T_p = 1238 \text{ MeV}$$

$$v = \frac{\sqrt{E_p^2 - m_p^2}}{E_p} = 0.65$$ not so small !!

2.2. Two particle system

System with particles, 1 and 2

$$p_1^\mu = (E_1, \boldsymbol{p}_1), \qquad E_1 = \sqrt{m_1^2 + \boldsymbol{p}_1^2}$$
$$p_2^\mu = (E_2, \boldsymbol{p}_2), \qquad E_2 = \sqrt{m_2^2 + \boldsymbol{p}_2^2}$$

Total 4-momentum

$$p^\mu = p_1^\mu + p_2^\mu$$

A Mandelstam variable

$$s = p^\mu p_\mu = (E_1 + E_2)^2 - (\boldsymbol{p}_1 + \boldsymbol{p}_2)^2$$

Lorentz invariant !

†) There are the 3 Mandelstam variables (s, t, u)

2.2.1 Center of mass (cm) frame

We attach $*$ on the quantities in the cm frame

- Definition

The total 3-momentum obeys

$$\boldsymbol{P}^* = \boldsymbol{p}_1^* + \boldsymbol{p}_2^* = 0$$

- Useful formulas

$$\begin{aligned} s &= (E_1^* + E_2^*)^2 - (\boldsymbol{p}_1^* + \boldsymbol{p}_2^*)^2 \\ &= (E_1^* + E_2^*)^2 = (E_{\text{total}}^*)^2 \end{aligned}$$

$$\begin{aligned} E_1^* &= \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}, \\ E_2^* &= \frac{s - m_1^2 + m_2^2}{2\sqrt{s}} \end{aligned}$$

[Just for fun]

$$|\boldsymbol{p}_1^*| = |\boldsymbol{p}_2^*| \\ = \frac{\sqrt{s^2 - 2s(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2}}{2\sqrt{s}}$$

R. Hagedron, *Relativistic Kinematics*,

The Benjamin/Cumming Pub. Co, INC (1963)

3. Cross Section

Experimental data are usually presented in terms of Cross Section

We observe

F : Number of incident particles
per unit time per unit surface
Flux (Strength of the incident beam)

ΔN : Number of particles getting
into the counter per unit time

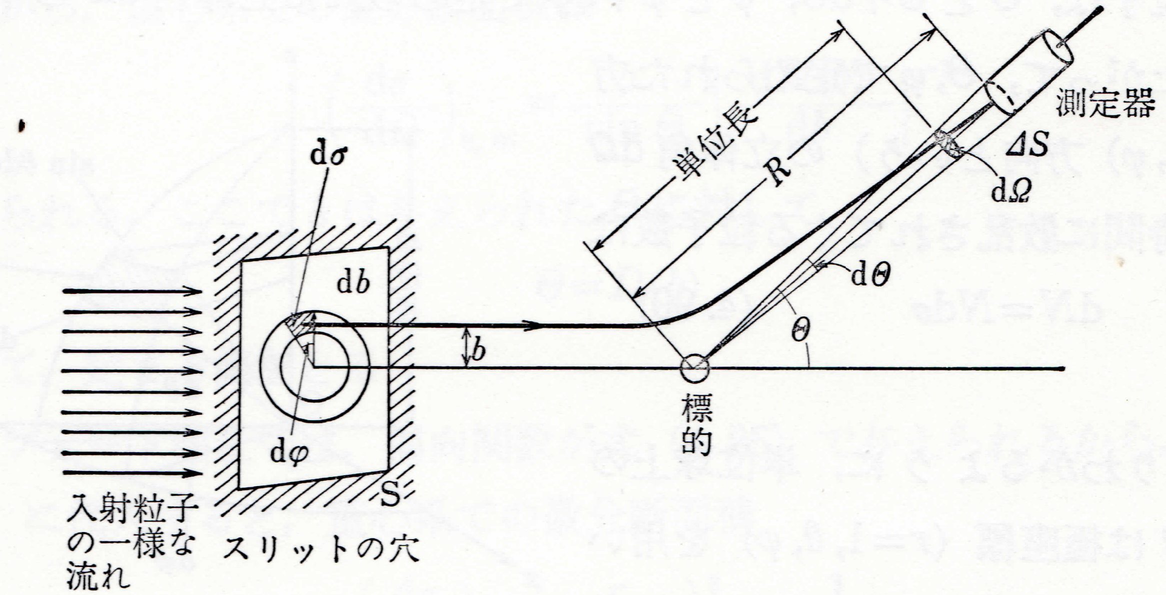


図 4.16 散乱微分断面積の説明

図 3 M. Ichimura, Mechanics, (1981) Asakura Pub. Co.

ΔN can be expressed as

$$\Delta N = C\frac{F\Delta S}{R^2} = CF\Delta\Omega$$

R : Distance between the target and the counter

ΔS : Area of the window of the counter

$\Delta\Omega$: Solid angle for the counter window

$$\Delta S = R^2\Delta\Omega$$

C represents the strength of the reaction independent of the experimental system

$$C = \frac{\Delta N}{F \Delta \Omega} \longrightarrow \qquad \frac{d\sigma}{d\Omega} = \lim_{\Delta\Omega \to 0} \frac{\Delta N}{F \Delta \Omega}$$

This is called differential cross section.

We will discuss various cross sections later.

[Comment] Dimension of C

$$[C] = \frac{[\Delta N]}{[F][\Delta \Omega]} = \frac{[T^{-1}]}{[T^{-1}L^{-2}][1]} = [L^2]$$

4. Formulas for Cross Sections

4.1 S matrix

State of the system

$$|i\rangle \quad \text{at } t = -\infty$$
$$|f\rangle \quad \text{at } t = \ \infty$$

Transition amplitude ($i \to f$)

$$S_{fi} = \langle f|S|i\rangle$$

This is called S matrix

[Just for fun]

A formal definition of S operator

(1) S is given by

the time developing unitary operator U as

$$S = \lim_{T\to\infty} U\left(\frac{T}{2}, -\frac{T}{2}\right)$$

where

$$U(t, t_0) = \mathrm{e}^{\mathrm{i}H_0 t}\mathrm{e}^{-\mathrm{i}H(t-t_0)}\mathrm{e}^{-\mathrm{i}H_0 t_0}$$

(2) Equivalently but more commonly by use of Möller operators $\Omega_\pm$

$$S = \Omega_-^\dagger \Omega_+$$

where

$$\Omega_+ = \lim_{t\to-\infty} \mathrm{e}^{\mathrm{i}Ht}\mathrm{e}^{-\mathrm{i}H_0 t}$$
$$\Omega_- = \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}Ht}\mathrm{e}^{-\mathrm{i}H_0 t}$$

4.2 T matrix

S matrix is written as

$$S_{fi} \equiv \delta_{fi} + \mathrm{i}(2\pi)^4\,\delta^{(4)}(p_f^\mu - p_i^\mu)\,T_{fi}$$
$$= \delta_{fi} + \mathrm{i}(2\pi)^4\delta(E_f - E_i)\,\delta^{(3)}(\boldsymbol{P}_f - \boldsymbol{P}_i)\,T_{fi}$$

E_i : Total energy of the initial state
E_f : Total energy of the final state
$\boldsymbol{P}_i$: Total momentum of the initial state
$\boldsymbol{P}_f$: Total momentum of the final state

T_{fi} : T-matrix

It involves at least one interaction.

4.3 Transition probability

- Transition probability

$$P(i \to f) = |S_{fi}|^2$$

For $|i\rangle \neq |f\rangle$

$$\begin{aligned} & P(i \to f) \\ & = [(2\pi)\delta(E_f - E_i)]^2[(2\pi)^3\delta^{(3)}(\boldsymbol{P}_f - \boldsymbol{P}_i)]^2 \\ & \times|T_{fi}|^2 \\ & = TV(2\pi)^4\delta(E_f - E_i)\delta^{(3)}(\boldsymbol{P}_f - \boldsymbol{P}_i)|T_{fi}|^2 \end{aligned}$$

V : Normalization volume ($V = 1$)

T : Elapsed time

Used a trick

$$\begin{aligned} & [(2\pi)\delta(E_f - E_i)]^2 \\ & = (2\pi)\delta(E_f - E_i)\int_{-T/2}^{T/2} \mathrm{e}^{\mathrm{i}(E_f - E_i)t}dt \\ & = T(2\pi)\delta(E_f - E_i) \end{aligned}$$

4.4 Transition rate

- Transition rate

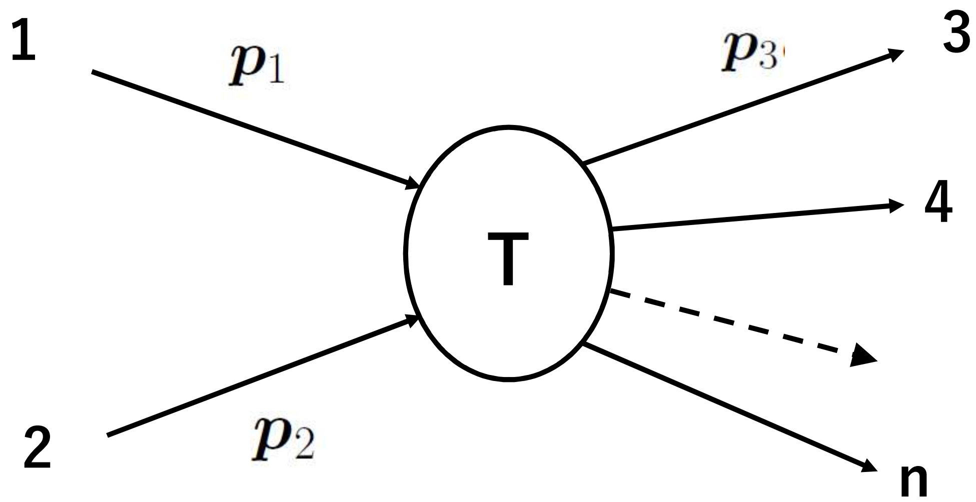
$$w_{fi} = \frac{P(i \to f)}{T}$$
$$= (2\pi)^4 \delta(E_f - E_i)\delta^{(3)}(\boldsymbol{P}_f - \boldsymbol{P}_i)|T_{fi}|^2$$

We set $V = 1$

4.5 Expression of the cross section

Consider the reaction

$$1 + 2 \to 3 + 4 + \cdots$$



$$E_i = E_1 + E_2, \quad E_f = E_3 + E_4 + \cdots$$

$$\boldsymbol{P}_i = \boldsymbol{p}_1 + \boldsymbol{p}_2, \quad \boldsymbol{P}_f = \boldsymbol{p}_3 + \boldsymbol{p}_4 + \cdots$$

$$E_j = \sqrt{m_j^2 + \boldsymbol{p}_j^2}, \quad (j = 1, 2, 3, 4, \cdots)$$

We observe the exit particles with accuracy

$$d^3\boldsymbol{p}_3 d^3\boldsymbol{p}_4 \cdots$$

in which number of states is

$$\frac{d^3\boldsymbol{p}_3}{(2\pi)^3}\frac{d^3\boldsymbol{p}_4}{(2\pi)^3}\cdots$$

Transition rate to the observed final states

$$dw_{fi} = (2\pi)^4\delta(E_f - E_i)\delta^{(3)}(\boldsymbol{P}_f - \boldsymbol{P}_i) \\ \times |T_{fi}|^2 \frac{d^3\boldsymbol{p}_3}{(2\pi)^3}\frac{d^3\boldsymbol{p}_4}{(2\pi)^3}\cdots$$

Cross section

$$d\sigma = \frac{dw_{fi}}{F}$$

with the flux

$$F = |\boldsymbol{v}_1 - \boldsymbol{v}_2| = v_{\mathrm{rel}}$$

Now we get

$$d\sigma = \frac{(2\pi)^4 \delta(E_f - E_i) \delta^{(3)}(\boldsymbol{P}_f - \boldsymbol{P}_i)}{v_{\mathrm{rel}}} \times |T_{fi}|^2 \frac{d^3\boldsymbol{p}_3}{(2\pi)^3} \frac{d^3\boldsymbol{p}_4}{(2\pi)^3} \cdots$$

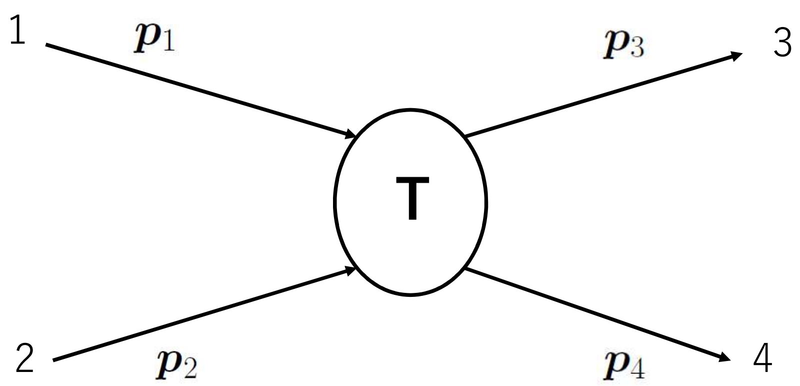
It is expressed by

- T-matrix,
- phase volume of the final states,
- relative velocity of the initial channel
- energy-momentum conservation condition

4.6 Two-body to two-body reactions

Reaction

$$1 + 2 \longrightarrow 3 + 4$$



Cross section

$$d\sigma = \frac{\delta(E_f - E_i)\delta^{(3)}(\boldsymbol{P}_f - \boldsymbol{P}_i)}{(2\pi)^2|\boldsymbol{v}_1 - \boldsymbol{v}_2|}|T_{fi}|^2 d^3\boldsymbol{p}_3 d^3\boldsymbol{p}_4$$

$$E_i = E_1 + E_2, \qquad E_f = E_3 + E_4$$

$$\boldsymbol{P}_i = \boldsymbol{p}_1 + \boldsymbol{p}_2, \qquad \boldsymbol{P}_f = \boldsymbol{p}_3 + \boldsymbol{p}_4$$

4.6.1 Center of mass frame

Attach $*$ on the quantities in the cm. frame

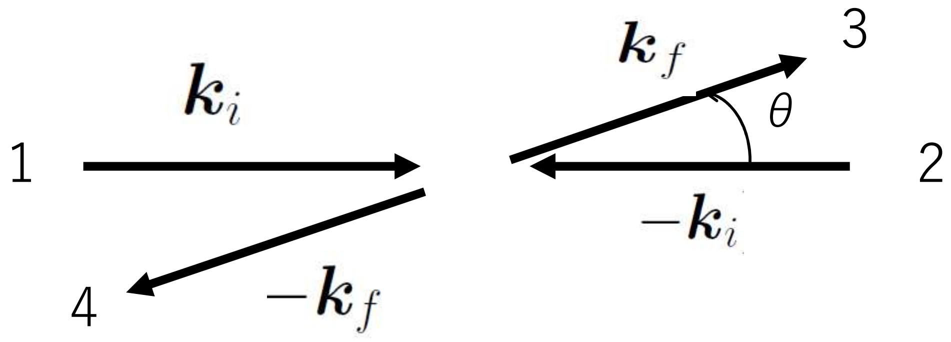


図 4 θ : Scattering angle (omitted $*$)

We write

$$\boldsymbol{k}_i = \boldsymbol{p}_1^* = -\boldsymbol{p}_2^*, \quad \boldsymbol{k}_f = \boldsymbol{p}_3^* = -\boldsymbol{p}_4^*$$

In this frame

$$\boldsymbol{P}_i^* = \boldsymbol{p}_1^* + \boldsymbol{p}_2^* = 0,$$
$$\sqrt{s} = E_i^* = E_1^* + E_2^*$$

We can calculate

$$\boldsymbol{v}_1^* - \boldsymbol{v}_2^* = \frac{\boldsymbol{k}_i}{E_1^*} - \frac{-\boldsymbol{k}_i}{E_2^*} = \frac{\boldsymbol{k}_i}{\mu_i}$$

with

$$\mu_i = \frac{E_1^* E_2^*}{E_1^* + E_2^*} = \frac{E_1^* E_2^*}{\sqrt{s}}$$

The Reduced energy of the incident channel.

The cross section becomes

$$d\sigma = \delta(E_3^* + E_4^* - \sqrt{s})\delta^{(3)}(\boldsymbol{p}_3^* + \boldsymbol{p}_4^*) \times \frac{1}{(2\pi)^2}\frac{\mu_i}{k_i}|T_{fi}|^2 d^3\boldsymbol{p}_3^* d^3\boldsymbol{p}_4^*$$

Integrate over $\boldsymbol{p}_4^*$, note $\boldsymbol{p}_3^* = \boldsymbol{k}_f$,

Use the energy-angle representation,

$$d^3\boldsymbol{k}_f = k_f E_3^* \, dE_3^* d\Omega$$

we get

$$d\sigma = \delta(E_3^* + E_4^* - \sqrt{s})dE_3^* \times \frac{\mu_i E_3^*}{(2\pi)^2}\frac{k_f}{k_i}|T_{fi}|^2 d\Omega$$

Noting

$$E_4^* = \sqrt{m_4^2 + k_f^2} = \sqrt{m_4^2 + (E_3^*)^2 - m_3^2}$$

we can rewrite

$$\delta(E_3^* + E_4^* - \sqrt{s})dE_3^* = \frac{\mu_f}{E_3^*}\delta(E_3^* - \bar{E}_3^*)dE_3^*$$

with

$$\mu_f = \frac{E_3^* E_4^*}{E_3^* + E_4^*} = \frac{E_3^*(\sqrt{s} - E_3^*)}{\sqrt{s}}$$

and

$$\bar{E}_3^* = \frac{s + m_3^2 - m_4^2}{2\sqrt{s}}$$

We used the delta function formula

$$\delta(f(x))dx = \sum_\alpha \frac{\delta(x - x^\alpha)}{|df(x)/dx|}dx$$

x^α : zero point of $f(x)$

Finally we get

● Double differential cross section

$$\frac{d^2\sigma}{dE_3^* d\Omega} = K|T_{fi}|^2 \delta(E_3^* - \bar{E}_3^*)$$

with the kinetic factor

$$K \equiv \frac{\mu_i \mu_f}{(2\pi)^2} \frac{k_f}{k_i},$$

the Reduced Energies

$$\mu_i = \frac{E_1^* E_2^*}{\sqrt{s}},$$

$$\mu_f = \frac{E_3^* E_4^*}{\sqrt{s}} = \frac{\bar{E}_3^*(\sqrt{s} - \bar{E}_3^*)}{\sqrt{s}}$$

and the ejectile energy

$$\bar{E}_3^* = \frac{s + m_3^2 - m_4^2}{2\sqrt{s}}$$

● Differential cross section

If the particle 4 is

- an elementary particle (no excited state),

or

- in the well isolated state,

we get the differential cross section by integrating over E_3^* as

$$\frac{d\sigma}{d\Omega} = K \; |T_{fi}|^2$$

[Just for fun]

In the textbooks of the relativity,
the T-matrix is written as

$$T_{fi} \equiv \frac{M_{fi}}{\sqrt{(2E_1)(2E_2)(2E_3)(2E_4)}}$$

and then the cross section is expressed as

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{k_f}{k_i} |M_{fi}|^2$$

4.6.2 Energy transfer expression

Double differential cross sections are often expressed with respect to the energy transfer to the target

$$\omega^* = E_1^* - E_3^*$$

Double differential cross section

$$\frac{d^2\sigma}{d\omega^* d\Omega} = K \, |T_{fi}|^2 \, \delta(\omega^* - \bar{\omega}^*)$$

with

$$\bar{\omega}^* = E_1^* - \bar{E}_3^* = \frac{m_1^2 - m_2^2 - m_3^2 + m_4^2}{2\sqrt{s}}$$

4.7 Non-relativistic formulas

- Reduced energy $\Longrightarrow$ Reduced mass

$$\mu_i = \frac{m_1 m_2}{m_1 + m_2}, \quad \mu_f = \frac{m_3 m_4}{m_3 + m_4}$$

- Total energy

$$E_1^* + E_2^* = m_1 + m_2 + T_i$$
$$E_3^* + E_4^* = m_3 + m_4 + T_f$$

- Kinetic energy of the relative motion

$$T_i = \frac{k_i^2}{2\mu_i}, \qquad T_f = \frac{k_f'^2}{2\mu_f}$$

- Reaction Q-value

$$Q = m_1 + m_2 - (m_3 + m_4)$$

Rewriting the infinitesimal phase volume as

$$d^3\boldsymbol{k}_f = k_f^2 dk_f d\Omega = k_f \mu_f dT_f d\Omega$$

and thus

$$\delta(E_1^* + E_2^* - E_3^* - E_4^*) d^3\boldsymbol{k}_f$$
$$= \mu_f k_f \delta(T_i + Q - T_f) dT_f d\Omega$$

we get

- **Double differential cross section**

$$\frac{d^2\sigma}{dT_f d\Omega} = K\, |T_{fi}|^2 \delta(T_i + Q - T_f)$$

- **Differential cross section**

$$\frac{d\sigma}{d\Omega} = K\, |T_{fi}|^2$$

5. Inclusive Cross Section

Return to the general cace

$$1 + 2 \to 3 + 4 + \cdots + n$$

5.1. Inclusive measurement

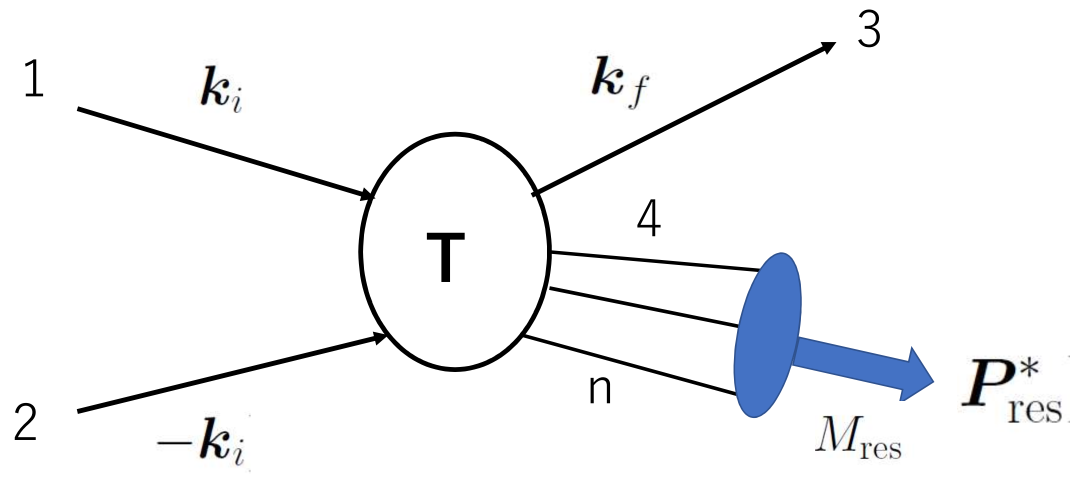
Detect only the particle 3

$$1 + 2 \to 3 + \text{anything}$$

Inclusive cross section

$$d\sigma = \frac{2\pi}{v_{\rm rel}} d^3\boldsymbol{p}_3 \int \delta(E_f - E_i)\delta^{(3)}(\boldsymbol{P}_f - \boldsymbol{P}_i)$$
$$\times |T_{fi}|^2 \frac{d^3\boldsymbol{p}_4}{(2\pi)^3} \cdots \frac{d^3\boldsymbol{p}_n}{(2\pi)^3}$$

5.2. Invariant mass



Introduce

- Total momentum of the residual system

$$\boldsymbol{P}^*_{\rm res} \equiv \sum_{j=4}^{n} \boldsymbol{p}_j$$

- Total energy of the residual system

$$E^*_{\rm res} \equiv \sum_{j=4}^{n} E_j$$

- Invariant mass of the residual system

$$M^2_{\rm res} \equiv E^{*2}_{\rm res} - \boldsymbol{P}^{*2}_{\rm res}$$

5.3 Formulas in the cm frame

In the cm frame

$$\boldsymbol{P}_f^* = \boldsymbol{p}_3^* + \boldsymbol{P}_{\rm res}^* = 0$$

As before we write

$$\boldsymbol{p}_3^* = \boldsymbol{k}_f$$

The cross section can be written as a similar form as the two-body to two-body reactions, by replacing

$$\boldsymbol{p}_4^* \to \boldsymbol{P}_{\rm res}^*, \quad E_4^* \to E_{\rm res}^*, \quad m_4 \to M_{\rm res}$$

.

● Double differential cross section

$$\frac{d^2\sigma}{d\omega^* d\Omega} = K \int |T_{fi}|^2 \, \delta(\omega^* - \bar{\omega}^*)$$

$$\times \; (2\pi)^3 \delta\left(\boldsymbol{k}_f + \boldsymbol{P}^*_{\rm res}\right) \frac{d^3\boldsymbol{p}^*_4}{(2\pi)^3} \cdots \frac{d^3\boldsymbol{p}^*_n}{(2\pi)^3}$$

with

$$\bar{\omega}^* = \frac{m_1^2 - m_2^2 - m_3^2 + M_{\rm res}^2}{2\sqrt{s}}$$

Ex. GTGR

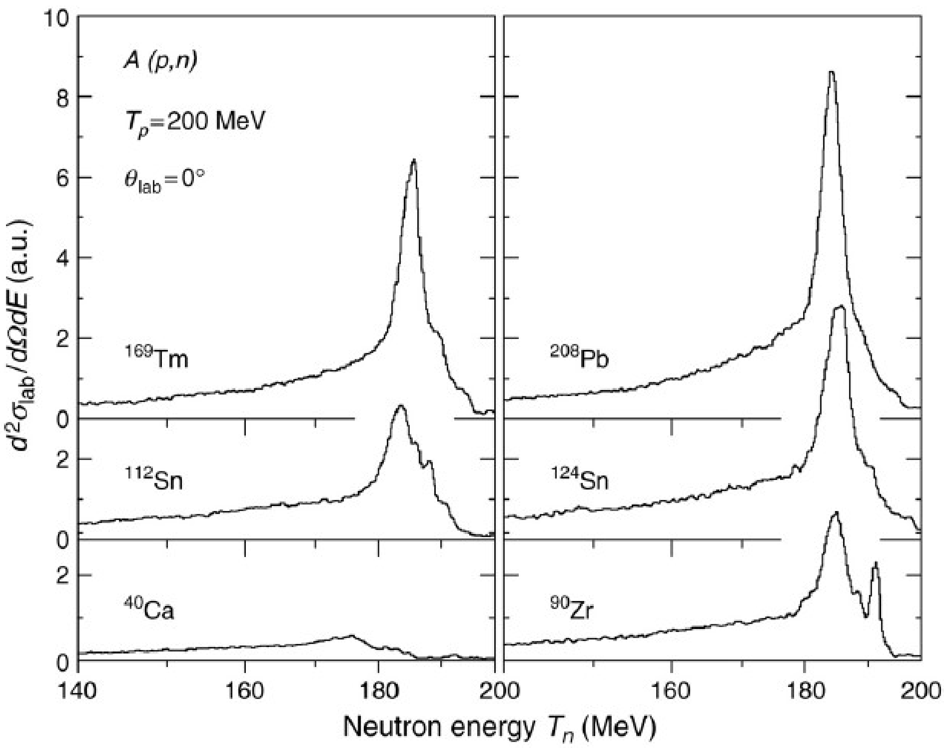


図 5　C. Gaarde et at., NP **A369**, 258(1981)

Using $\boldsymbol{P}^*_{\rm res} = \Sigma_{j=4}^n \boldsymbol{p}_j = -\boldsymbol{k}_f$, we may write

$$\int \delta\left(\boldsymbol{k}_f + \boldsymbol{P}^*_{\rm res}\right) d^3\boldsymbol{p}^*_4 \cdots d^3\boldsymbol{p}^*_n \cdots$$
$$= \int d^3\boldsymbol{\kappa}_1 \cdots d^3\boldsymbol{\kappa}_{n-4} \cdots$$

where $\boldsymbol{\kappa}_1 \cdots \boldsymbol{\kappa}_{n-4}$ are suitably chosen internal momenta of the residual system.

- **Double differential cross section**

$$\frac{d^2\sigma}{d\omega^* d\Omega}$$
$$= K \int |T_{fi}|^2\, \delta(\omega^* - \bar{\omega}^*) \frac{d^3\boldsymbol{\kappa}_1}{(2\pi)^3} \cdots \frac{d^3\boldsymbol{\kappa}_{n-3}}{(2\pi)^3}$$

How to treat the integral $\int d^3\boldsymbol{\kappa}_1 \cdots d^3\boldsymbol{\kappa}_{n-4}$ is a main subject of this lecture

Other cross sections found in the literature

- **Angular distribution**

Angular distribution at ω^*

$$\frac{d\sigma}{d\Omega} = \int_{\omega^*-\delta\epsilon}^{\omega^*+\delta\epsilon} \left(\frac{d^2\sigma}{d\omega^* d\Omega} \right) d\omega^*$$

- **Angle integrated energy differential cross section**

$$\frac{d\sigma}{d\omega^*} = \int \left(\frac{d^2\sigma}{d\omega_3^* d\Omega} \right) d\Omega$$

6. Frame Transformation

● Experiments are usually carried out in the laboratory (lab) frame, where

$$\boldsymbol{p}_2^{\mathrm{lab}} = 0$$

● Theoretical calculations are usually done in the cm frame, where

$$\boldsymbol{p}_1^* + \boldsymbol{p}_2^* = 0$$

We need the transformation formulas from the cm frame to the lab frame or vice versa.

6.1 Relativistic kinematics

Energy of the incident particle is usually written by the incident kinetic energy T_1 as

$$E_1^{\mathrm{lab}} = m_1 + T_1$$

thus

$$s = (m_1 + m_2)^2 + 2m_2T_1$$

● Lorentz transformation parameters, (β, γ)

$$\beta = V = \frac{|\boldsymbol{p}_1^{\mathrm{lab}}|}{E_1^{\mathrm{lab}} + E_2^{\mathrm{lab}}} = \frac{\sqrt{T_1(T_1 + 2m_1)}}{m_1 + m_2 + T_1}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{m_1 + m_2 + T_1}{\sqrt{s}}$$

V : Velocity of the center of mass

[Exercise] Prove the above formulas.

● Lorentz transformation

The energy-momentum of the ejectile (particle 3)

$$\begin{aligned} p_3^{\text{lab}} \cos\theta_{\text{lab}} &= \gamma(p_3^* \cos\theta + \beta E_3^*) \\ p_3^{\text{lab}} \sin\theta_{\text{lab}} &= p_3^* \sin\theta \\ E_3^{\text{lab}} &= \gamma(E_3^* + \beta p_3^* \cos\theta) \end{aligned}$$

From them, we get

(1) **Scattering angle**

$$\tan\theta_{\text{lab}} = \frac{\sin\theta}{\gamma(\cos\theta + \alpha)}$$

with

$$\alpha = \frac{V}{v_3^*}, \qquad v_3^* = \frac{p_3^*}{E_3^*}$$

v_3^* : velocity of the ejectile (particle 3)

(2) **Double differential cross section**

$$\left(\frac{d^2\sigma}{dE_3^* d\Omega}\right) = \left(\frac{d^2\sigma}{dE_3^{\text{lab}} d\Omega_{\text{lab}}}\right) \frac{\partial(E_3^{\text{lab}}, \Omega_{\text{lab}})}{\partial(E_3^*, \Omega)}$$

$$\frac{\partial(E_3^{\text{lab}}, \Omega_{\text{lab}})}{\partial(E_3^*, \Omega)} = \left| \begin{matrix} \frac{\partial E_3^{\text{lab}}}{\partial E_3^*}, & \frac{\partial E_3^{\text{lab}}}{\partial \cos\theta} \\ \frac{\partial \cos\theta}{\partial E_3^*}, & \frac{\partial \cos\theta_{\text{lab}}}{\partial \cos\theta} \end{matrix} \right|$$

$$= \frac{1}{(\gamma^2(\cos\theta + \alpha)^2 + \sin^2\theta)^{1/2}} = \frac{\sin\theta_{\text{lab}}}{\sin\theta}$$

thus

$$\left(\frac{d^2\sigma}{dE_3^* d\Omega}\right) = \frac{\sin\theta_{\text{lab}}}{\sin\theta} \left(\frac{d^2\sigma}{dE_3^{\text{lab}} d\Omega_{\text{lab}}}\right)$$

or

$$\left(\frac{d^2\sigma}{d\omega^* d\Omega}\right) = \frac{\sin\theta_{\text{lab}}}{\sin\theta} \left(\frac{d^2\sigma}{d\omega^{\text{lab}} d\Omega_{\text{lab}}}\right)$$

For the case of final two elementary particles

Differential cross section

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega_{\text{lab}}}\right)\frac{d\Omega_{\text{lab}}}{d\Omega}$$

$$\frac{d\Omega_{\text{lab}}}{d\Omega} = \frac{\gamma(1+\alpha\cos\theta)}{(\gamma^2(\cos\theta+\alpha)^2+\sin^2\theta)^{3/2}}$$

6.2 Non-relativistic kinematics

Velocity of the center of mass in lab frame

$$V = \frac{m_1}{m_1 + m_2} v_1^{\mathrm{lab}}$$

Galilei transformation

$$\begin{aligned} v_3^{\mathrm{lab}} \cos\theta_{\mathrm{lab}} &= v_3^* \cos\theta + V \\ v_3^{\mathrm{lab}} \sin\theta_{\mathrm{lab}} &= v_3^* \sin\theta \end{aligned}$$

Just set $\gamma = 1$ in the relativistic formula,

- **Scattering angle**

$$\tan\theta_{\mathrm{lab}} = \frac{\sin\theta}{\cos\theta + \alpha}, \quad \alpha = \frac{V}{v_3^*}$$

- **Differential cross section**

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega_{\mathrm{lab}}}\right) \frac{1 + \alpha\cos\theta}{(1 + 2\alpha\cos\theta + \alpha^2)^{3/2}}$$

● Double differential cross section

We write the energies as

$$E_3^{\text{lab}} = m_3 + \frac{(p_3^{\text{lab}})^2}{2m_3} = m_3 + T_3^{\text{lab}}$$

$$E_3^* = m_3 + \frac{k_f^2}{2m_3} = m_3 + T_3^*$$

$$T_f = \frac{k_f^2}{2\mu_f} = \frac{m_3 + m_4}{m_4} T_3^*$$

$$\left(\frac{d^2\sigma}{dT_f d\Omega}\right) = \left(\frac{d^2\sigma}{dT_3^{\text{lab}} d\Omega_{\text{lab}}}\right) \frac{\partial(T_3^{\text{lab}}, \Omega_{\text{lab}})}{\partial(T_3^*, \Omega)} \frac{dT_3^*}{dT_f}$$

we get

$$\left(\frac{d^2\sigma}{dT_f d\Omega}\right) = \left(\frac{d^2\sigma}{dT_3^{\text{lab}} d\Omega_{\text{lab}}}\right) \frac{m_4}{m_3 + m_4} \frac{\sin\theta_{\text{lab}}}{\sin\theta}$$

[Just for fun]

Calculation of $\dfrac{\partial(T_3^{\text{lab}}, \Omega_{\text{lab}})}{\partial(T_3^*, \Omega)}$

From the velocity-angle relations

$$v_3^{\text{lab}} \cos\theta_{\text{lab}} = v_3^* \cos\theta + V$$

$$v_3^{\text{lab}} \sin\theta_{\text{lab}} = v_3^* \sin\theta$$

$$\tan\theta_{\text{lab}} = \frac{\sin\theta}{\cos\theta + \alpha}$$

we get

$$\cos\theta_{\text{lab}} = \left(\frac{1}{1+\tan^2\theta_{\text{lab}}}\right)^{1/2} = \frac{\cos\theta + \alpha}{(1 + 2\alpha\cos\theta + \alpha^2)^{1/2}}$$

$$(v_3^{\text{lab}})^2 = (v_3^*)^2(1 + 2\alpha\cos\theta + \alpha^2)$$

thus

$$T_3^{\text{lab}} = T_3^*(1 + 2\alpha\cos\theta + \alpha^2)$$

The velocity ratio α is given by

$$\alpha = \frac{V}{v_3^*} = V\sqrt{\frac{m_3}{2T_3^*}}$$

thus

$$\frac{\partial\alpha}{\partial T_3^*} = -\frac{\alpha}{2T_3^*}, \qquad \frac{\partial\alpha}{\partial\cos\theta} = 0$$

Now we get

$$
\begin{aligned}
\frac{\partial T_3^{\rm lab}}{\partial T_3^*} &= (1+2\alpha\cos\theta+\alpha^2) + 2T_3^*(\cos\theta+\alpha)\frac{\partial\alpha}{\partial T_3^*} \\
&= 1+\alpha\cos\theta \\
\frac{\partial T_3^{\rm lab}}{\partial\cos\theta} &= 2\alpha T_3^* \\
\frac{\partial\cos\theta_{\rm lab}}{\partial T_3^*} &= \frac{\partial\cos\theta_{\rm lab}}{\partial\alpha}\frac{\partial\alpha}{\partial T_3^*} \\
&= -\frac{\alpha}{2T_{\rm cm}^b}\frac{\sin^2\theta_{\rm cm}}{(1+2\alpha\cos\theta_{\rm cm}+\alpha^2)^{3/2}} \\
\frac{\partial\cos\theta_{\rm lab}}{\partial\cos\theta} &= \frac{1+\alpha\cos\theta}{(1+2\alpha\cos\theta+\alpha^2)^{3/2}}
\end{aligned}
$$

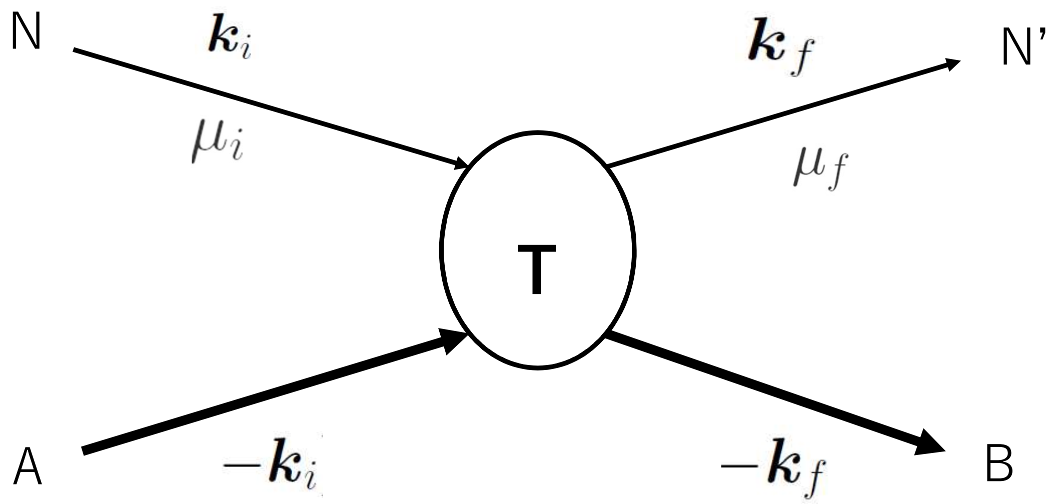
Using these formulas, we finally obtained the Jacobian

$$
\begin{aligned}
\frac{\partial(T_3^{\rm lab},\Omega_{\rm lab})}{\partial(T_3^*,\Omega)} = \frac{\partial(T_3^{\rm lab},\cos\theta_{\rm lab})}{\partial(T_3^*,\cos\theta)} &= \left|\begin{array}{cc} \frac{\partial T_3^{\rm lab}}{\partial T_3^*}, & \frac{\partial T_3^{\rm lab}}{\partial\cos\theta} \\ \frac{\partial\cos\theta_{\rm lab}}{\partial T_3^*}, & \frac{\partial\cos\theta_{\rm lab}}{\partial\cos\theta} \end{array}\right| \\
&= \frac{1}{(1+2\alpha\cos\theta+\alpha^2)^{1/2}} = \frac{v_3^*}{v_3^{\rm lab}} = \frac{\sin\theta_{\rm lab}}{\sin\theta}
\end{aligned}
$$

7. Spin Observables

When the projectile and the ejectile have spins, and their directions are observed, the data present us very powerful and fruitful information to study the nuclear structure.
As an example, I want to refer spin observables in the nucleon scatterings

$$A(N, N')B$$



7.1 Review about spin operators

Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Spin operator

$$\boldsymbol{s} = \frac{1}{2}\boldsymbol{\sigma}$$

Useful formulas for $\sigma_a \;\; (a = x, y, z)$

$$\sigma_a \sigma_a = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_a \sigma_b = \mathrm{i}\epsilon_{abc}\sigma_c$$

$$\mathrm{Tr}\left[\sigma_a\right] = 0, \quad \mathrm{Tr}\left[\sigma_a \sigma_b\right] = 2\delta_{ab}$$

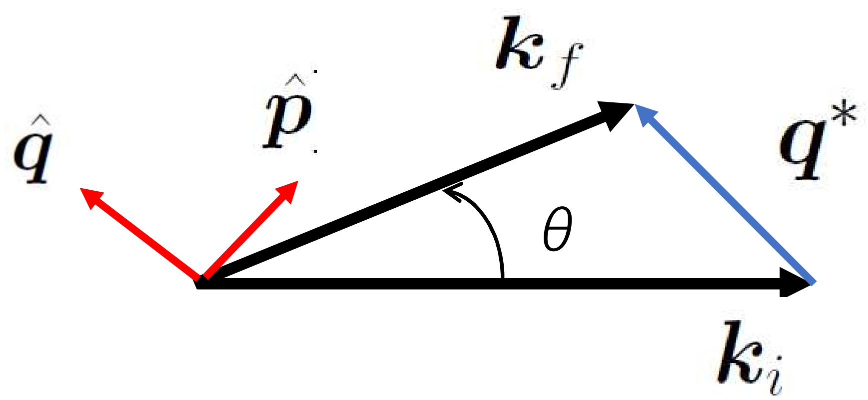
7.2. Coordinate systems

The following coordinate system is often used in the theoretical analysises

- **$[\hat{\boldsymbol{q}},\ \hat{\boldsymbol{n}},\ \hat{\boldsymbol{p}}]$ system**

$$\hat{\boldsymbol{q}} = \frac{\boldsymbol{q}^*}{\mid \boldsymbol{q}^* \mid}, \quad \hat{\boldsymbol{n}} = \frac{\boldsymbol{k}_i \times \boldsymbol{k}_f}{|\boldsymbol{k}_i \times \boldsymbol{k}_f|}, \quad \hat{\boldsymbol{p}} = \hat{\boldsymbol{q}} \times \hat{\boldsymbol{n}}$$

Their directions are denoted by (q, n, p).



[Just for fun]

(1) $[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$ system

$$\hat{\boldsymbol{z}} = \frac{\boldsymbol{k}_i}{|\boldsymbol{k}_i|}, \qquad \hat{\boldsymbol{y}} = \hat{\boldsymbol{n}}, \qquad \hat{\boldsymbol{x}} = \hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}} \ ,$$

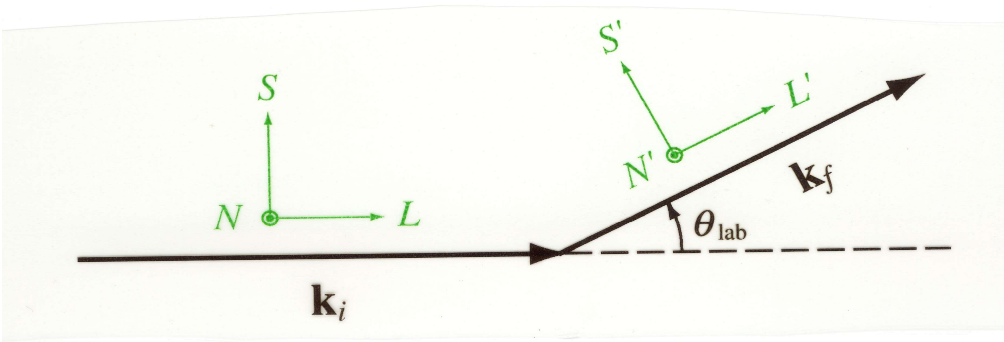
This is used in the numerical calculation.

(2) $[\boldsymbol{S}, \boldsymbol{N}, \boldsymbol{L}]$, and $[\boldsymbol{S}', \boldsymbol{N}', \boldsymbol{L}']$ systems

$$\hat{\boldsymbol{S}} = \hat{\boldsymbol{x}}, \quad \hat{\boldsymbol{N}} = \hat{\boldsymbol{y}}, \quad \hat{\boldsymbol{L}} = \hat{\boldsymbol{z}}$$

$$\hat{\boldsymbol{N}}' = \hat{\boldsymbol{N}} \ , \quad \hat{\boldsymbol{L}}' = \frac{\boldsymbol{k}_f^{\rm lab}}{|\boldsymbol{k}_f^{\rm lab}|} \ , \quad \hat{\boldsymbol{S}}' = \hat{\boldsymbol{N}}' \times \hat{\boldsymbol{L}}' \ .$$

Used for experimental data in the lab. frame



7.3. T-matrix

T-matrix is specified more explicitly as

$$
\begin{aligned}
T_{fi} &= \langle \boldsymbol{k}_f \mu_f \mathrm{N}', \Phi_B | T | \boldsymbol{k}_i \mu_i \mathrm{N}, \Phi_A \rangle \\
&= \left(\langle \boldsymbol{k}_f \mathrm{N}', \Phi_B | T | \boldsymbol{k}_i, \mathrm{N}, \Phi_A \rangle \right)_{\mu_f, \mu_i}
\end{aligned}
$$

This is a 2 x 2 matrix with respect to (μ_f, μ_i)

It is generally written as

$$T_{fi} = T_0 + T_n \sigma_n + T_q \sigma_q + T_p \sigma_p$$

σ_i's denote the spin operator for N (N').

T_i's abbreviate the marix element.

$$T_i = \langle \boldsymbol{k}_f, \mathrm{N}', \Phi_B | T_i | \boldsymbol{k}_i, \mathrm{N}, \Phi_A \rangle$$

7.4. Unpolarized cross section

- **Unpolarized differential cross section**

$$I = \frac{d^2\sigma}{d\omega^* d\Omega}$$
$$= K \sum_{\mu_f} \frac{1}{2} \sum_{\mu_i} \left| (T)_{\mu_f \mu_i} \right|^2 = \frac{K}{2} \mathrm{Tr} \left[T^\dagger T \right]$$
$$= K \left[T_0^\dagger T_0 + T_n^\dagger T_n + T_q^\dagger T_q + T_p^\dagger T_p \right]$$

Tr : Trace with respect to the nucleon spin.

Define

Polarized Cross Sections ID_i

$$ID_i = K \left[T_i^\dagger T_i \right], \quad (i = 0, q, n, p)$$

Cross section is expressed as

$$\frac{d^2\sigma}{d\omega^* d\Omega} = ID_0 + ID_n + ID_q + ID_p$$

7.5. Spin observables

- **Polarization**

$$P_y = \frac{\mathrm{Tr}[T^\dagger \sigma_y T]}{\mathrm{Tr}[T^\dagger T]} \ ,$$

- **Analyzing power**

$$A_y = \frac{\mathrm{Tr}[T^\dagger T \sigma_y]}{\mathrm{Tr}[T^\dagger T]}$$

- **Polarization transfer coefficients**

$$D_{ij} = \frac{\mathrm{Tr}[T^\dagger \sigma_i T \sigma_j]}{\mathrm{Tr}[T^\dagger T]}, \quad (i, j = p, q, n)$$

Refer Wakasa-san' talk about the definitions and the measurements of P_y, A_y, D_{ij}

- **Polarized cross sections**

We obtain ID_i by the combination of D_{ij} as

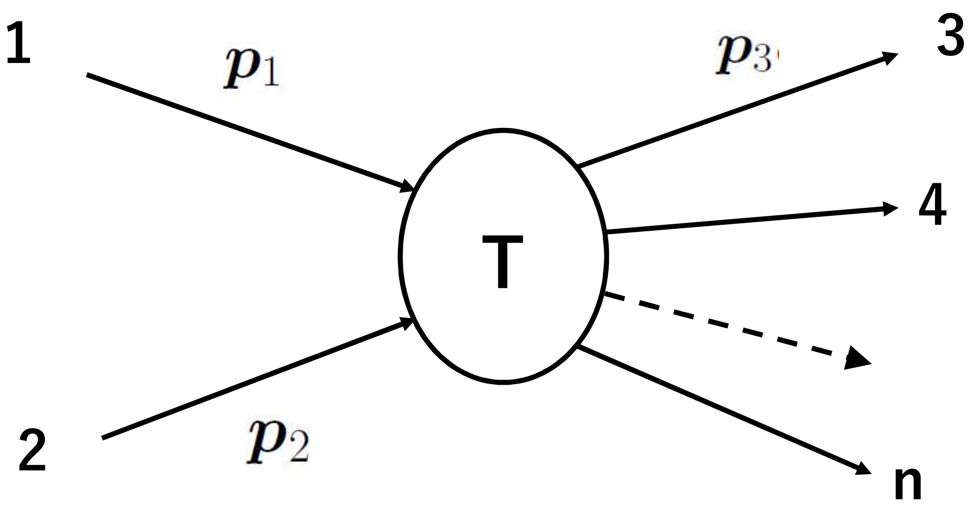
$$ID_0 = \frac{I}{4}[1 + D_{nn} + D_{qq} + D_{pp}] = K[T_0^\dagger T_0]$$
$$ID_n = \frac{I}{4}[1 + D_{nn} - D_{qq} - D_{pp}] = K[T_n^\dagger T_n]$$
$$ID_q = \frac{I}{4}[1 - D_{nn} + D_{qq} - D_{pp}] = K[T_q^\dagger T_q]$$
$$ID_p = \frac{I}{4}[1 - D_{nn} - D_{qq} + D_{pp}] = K[T_p^\dagger T_p]$$

They are very useful to extract spin-longitudinal and spin-transverse responses.

See Wakasa-san's lecture

8. Summary

We considered the reactions

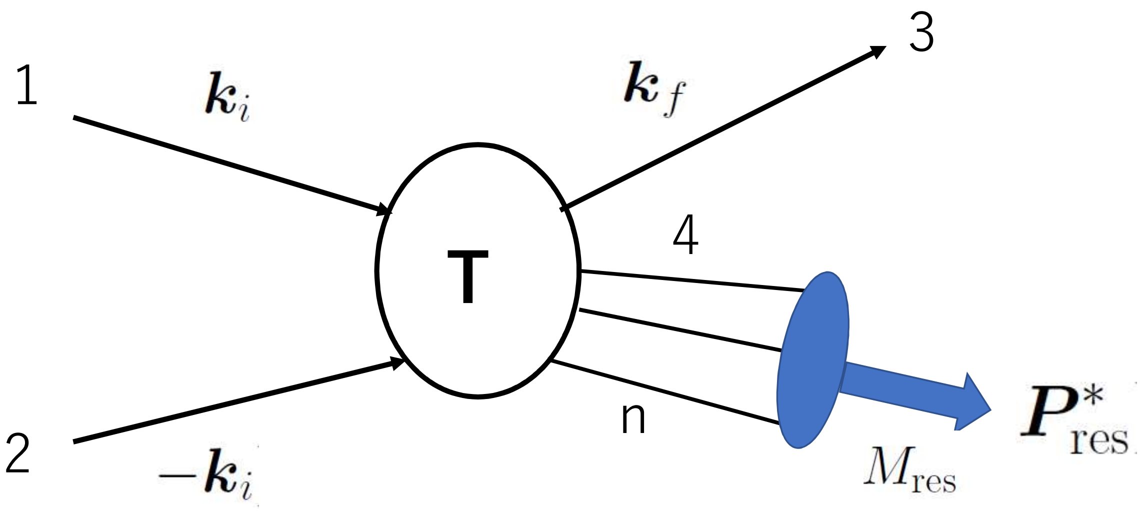


- Cross section

$$d\sigma = \frac{(2\pi)^4 \delta(E_f - E_i)\delta^{(3)}(\boldsymbol{P}_f - \boldsymbol{P}_i)}{v_{\rm rel}}$$

$$\times\ |T_{fi}|^2 \frac{d^3\boldsymbol{p}_3}{(2\pi)^3}\frac{d^3\boldsymbol{p}_4}{(2\pi)^3}\cdots$$

- Inclusive cross section in the cm frame



$$\frac{d^2\sigma}{d\omega^* d\Omega} = K \int |T_{fi}|^2 \, \delta(\omega^* - \bar{\omega}^*)$$

$$\times \, (2\pi)^3 \delta\left(\boldsymbol{k}_f + \boldsymbol{P}^*_{\rm res}\right) \frac{d^3\boldsymbol{p}^*_4}{(2\pi)^3} \cdots \frac{d^3\boldsymbol{p}^*_n}{(2\pi)^3}$$

$$\bar{\omega}^* = \frac{m_1^2 - m_2^2 - m_3^2 + M_{\rm res}^2}{2\sqrt{s}}$$

$$M_{\rm res}^2 = (\sum_{j=4}^{n} E_j)^2 - (\sum_{j=4}^{n} \boldsymbol{p}_j)^2$$

- Frame transformation

Double differential cross section

$$\left(\frac{d^2\sigma}{d\omega^* d\Omega}\right) = \frac{\sin\theta_{\rm lab}}{\sin\theta}\left(\frac{d^2\sigma}{d\omega^{\rm lab} d\Omega_{\rm lab}}\right)$$

Differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{\gamma(1+\alpha\cos\theta)}{(\gamma^2(\cos\theta+\alpha)^2+\sin^2\theta)^{3/2}}\left(\frac{d\sigma}{d\Omega_{\rm lab}}\right)$$

- Spin observables in (N, N') reaction

T-matrix

$$T_{fi} = T_0 + T_n\sigma_n + T_q\sigma_q + T_p\sigma_p$$

Unpolarized cross section

$$I = \frac{d^2\sigma}{d\omega^* d\Omega} = \frac{K}{2}\mathrm{Tr}\left[T^\dagger T\right]$$

Polarized cross sections

$$ID_0 = \frac{I}{4}[1 + D_{nn} + D_{qq} + D_{pp}] = K[T_0^\dagger T_0]$$
$$ID_n = \frac{I}{4}[1 + D_{nn} - D_{qq} - D_{pp}] = K[T_n^\dagger T_n]$$
$$ID_q = \frac{I}{4}[1 - D_{nn} + D_{qq} - D_{pp}] = K[T_q^\dagger T_q]$$
$$ID_p = \frac{I}{4}[1 - D_{nn} - D_{qq} + D_{pp}] = K[T_p^\dagger T_p]$$