Nuclear Direct Reactions to Continuum 3

– How to get Nuclear Structure Information –

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V. DWBA

1. Simple example
2. (N, N') reaction

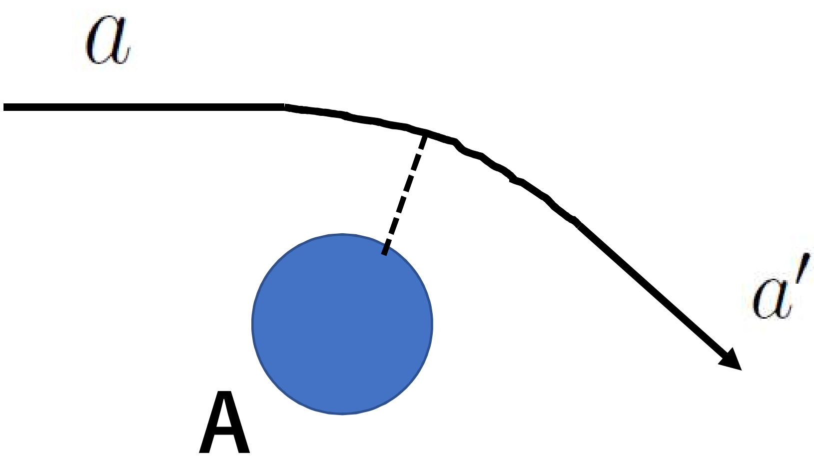
VI DWIA

1. Impulse Approximation
2. Free NN t-matrix
3. Off-energy-shell Extrapolation
4. Frame Transformation
5. Local Potential Approximation
6. PWIA

V. DWBA

Let's be more realistic. Consider **DWBA**

Distorted Wave Born Approximation



$a(a')$ is influenced by the mean field produced by the target (residual nucleus) A.

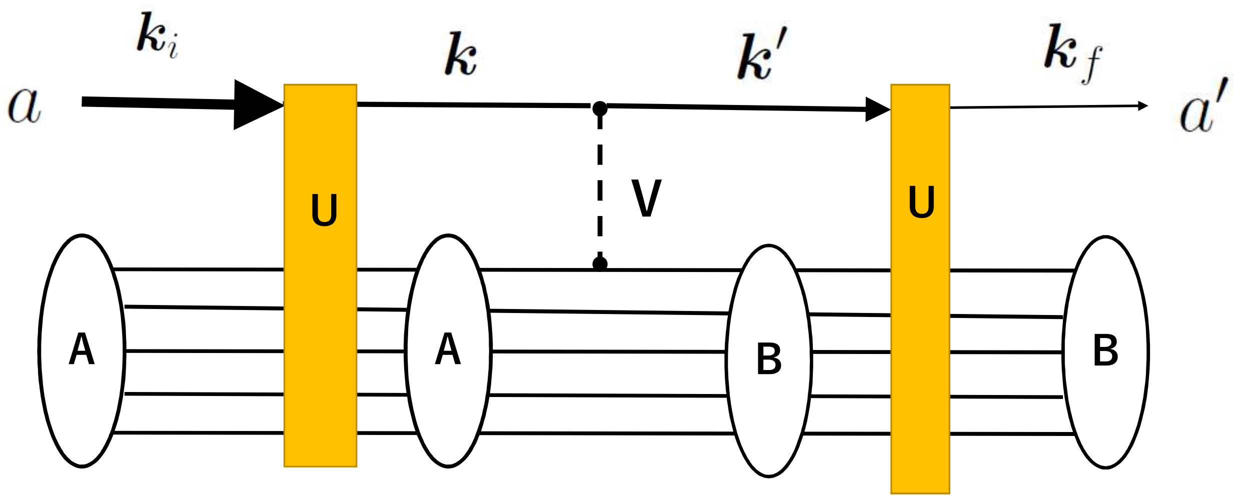
Distortion and Absorption

1. Simple example

Let us consider a simple example

$$A(a, a')B$$

$a,\ a'$: structureless point particles.



The mean field U is complex,

called Optical potential.

At the transition,

momentum change is $\boldsymbol{k} \longrightarrow \boldsymbol{k}'$,

not observed change $\boldsymbol{k}_i \longrightarrow \boldsymbol{k}_f$

● Distorted waves

The wave functions of a and a' are not plane waves any more.
they are expressed by the distorted waves, which are subject to

$$\left(-\frac{\boldsymbol{\nabla}^2}{2\mu_i}+U(\boldsymbol{r})\right)\chi^{(+)}_{\boldsymbol{k}_i}(\boldsymbol{r})=\frac{k_i^2}{2\mu_i}\,\chi^{(+)}_{\boldsymbol{k}_i}(\boldsymbol{r})$$

$$\left(-\frac{\boldsymbol{\nabla}^2}{2\mu_f}+U^*(\boldsymbol{r})\right)\chi^{(-)}_{\boldsymbol{k}_f}(\boldsymbol{r})=\frac{k_f^2}{2\mu_f}\,\chi^{(-)}_{\boldsymbol{k}_f}(\boldsymbol{r})$$

$(+)$: Outgoing boundary condition
$(-)$: Incoming boundary condition

● T-matrix in DWBA

T-marix is given by

$$T_{fi} = \langle \chi^{(-)}_{\boldsymbol{k}_f}(\boldsymbol{r}_0)\, \Phi_B | V | \Phi_A\, \chi^{(+)}_{\boldsymbol{k}_i}(\boldsymbol{r}_0) \rangle$$

● Interaction

We write the interaction as

$$V = \sum_{k \in A} V(\boldsymbol{r}_0 - \boldsymbol{r}_k) = \int d^3\boldsymbol{r} V(\boldsymbol{r}_0 - \boldsymbol{r})\rho(\boldsymbol{r})$$

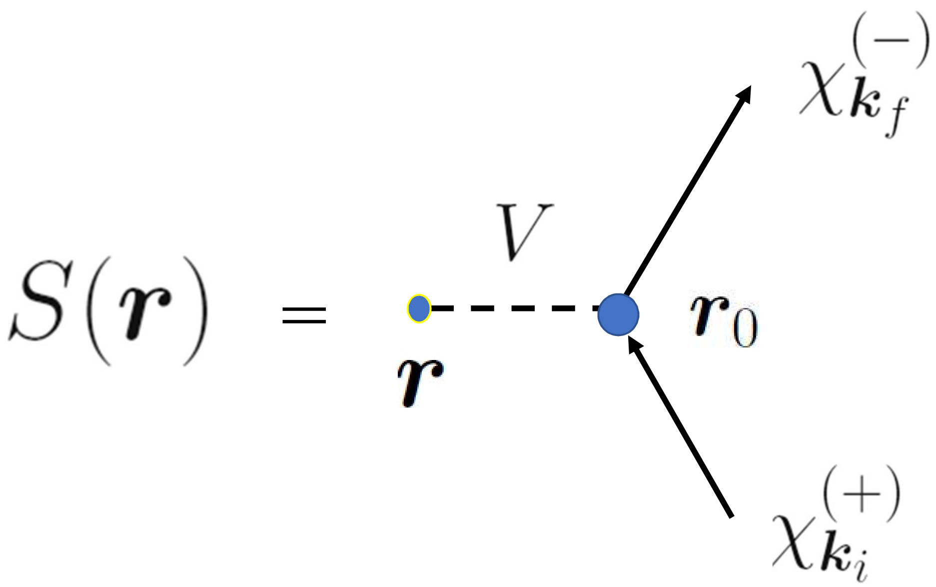
with the density operator

$$\rho(\boldsymbol{r}) = \sum_{k \in A} \delta(\boldsymbol{r} - \boldsymbol{r}_k)$$

● Outer Impulse Function

We introduce **Outer impulse function**

$$S(\boldsymbol{r}) \equiv \langle \chi_{\boldsymbol{k}_f}^{(-)}(\boldsymbol{r}_0)|V(\boldsymbol{r}_0 - \boldsymbol{r})|\chi_{\boldsymbol{k}_i}^{(+)}(\boldsymbol{r}_0)\rangle$$



● Transition form factor

Define the transition form factor
in the coordinate space

$$F_{BA}(\boldsymbol{r}) = \langle \Phi_B|\rho(\boldsymbol{r})|\Phi_A\rangle$$

● T-matrix

The T-matrix becomes

$$T_{fi} = \int d^3\boldsymbol{r}\ S(\boldsymbol{r})\ F_{BA}(\boldsymbol{r})$$

● Differential cross section

When B is an isolated discrete state, the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = K\ \int d^3\boldsymbol{r}' \int d^3\boldsymbol{r} \\ \times S^*(\boldsymbol{r}')\left[F^*_{BA}(\boldsymbol{r}')F_{BA}(\boldsymbol{r})\right]S(\boldsymbol{r})$$

$[F^*_{BA}(\boldsymbol{r}')F_{BA}(\boldsymbol{r})]$: structure part

[Comments]

(1) Transition form factor in PWBA.

$$\tilde{F}_{BA}(\boldsymbol{q}^*) = \langle \Phi_B | \sum_k \mathrm{e}^{-\mathrm{i}\boldsymbol{q}^*\cdot\boldsymbol{r}_k} | \Phi_A \rangle$$
$$= \int d^3\boldsymbol{r} F_{BA}(\boldsymbol{r}) \mathrm{e}^{-\mathrm{i}\boldsymbol{q}^*\cdot\boldsymbol{r}}$$

This is the Fourier transform of $F_{BA}(\boldsymbol{r})$.

(2) T-matrix in PWBA

$$T_{fi} = \tilde{V}(\boldsymbol{q}^*)\ F_{BA}(\boldsymbol{q}^*)$$

while T-matrix in DWBA

$$T_{fi} = \int d^3\boldsymbol{r}\ S(\boldsymbol{r})\ F_{BA}(\boldsymbol{r})$$

No more of the factorized form.

(3) Cross section in PWBA

$$\frac{d\sigma}{d\Omega} = K\ |V(\boldsymbol{q}^*)|^2\ |F_{BA}(\boldsymbol{q}^*)|^2$$

while in DWBA

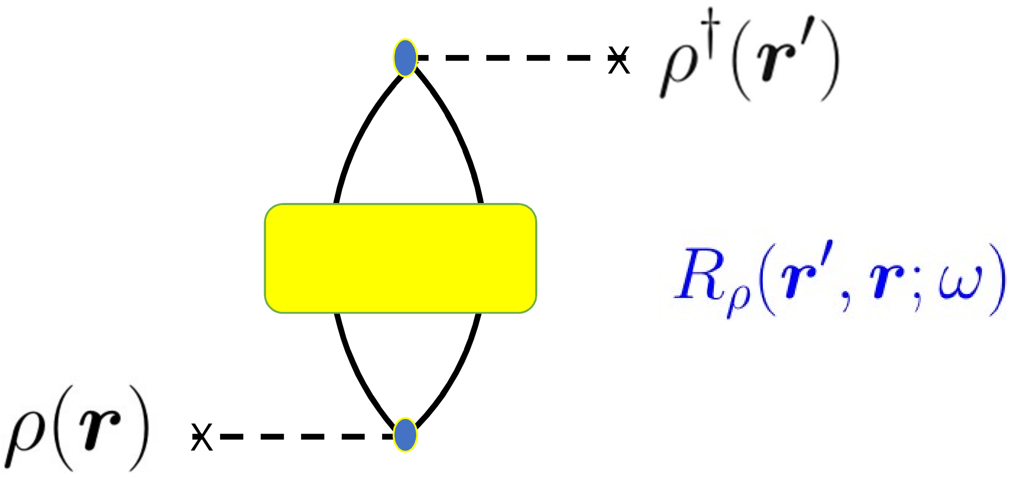
$$\frac{d\sigma}{d\Omega} = K\ \int d^3\boldsymbol{r}' \int d^3\boldsymbol{r}$$
$$\times\ S^*(\boldsymbol{r}')\,[F^*_{BA}(\boldsymbol{r}')F_{BA}(\boldsymbol{r})]\,S(\boldsymbol{r})$$

No more of the factorized form.

● Response Function

Define Response Function in $\boldsymbol{r}$ space

$$
\begin{aligned}
& R_\rho(\boldsymbol{r}', \boldsymbol{r}; \omega) \\
\equiv & \sum_X \langle \Phi_A | \rho^\dagger(\boldsymbol{r}') | \Phi_X \rangle \delta(\omega - E_x^X) \langle \Phi_X | \rho(\boldsymbol{r}) | \Phi_A \rangle \\
= & -\frac{1}{\pi} \mathrm{Im} \langle \Phi_A | \rho^\dagger(\boldsymbol{r}') \, \frac{1}{\omega - H_A + \mathrm{i}\delta} \, \rho(\boldsymbol{r}) \, | \Phi_A \rangle
\end{aligned}
$$



Purely nuclear structure quantity.

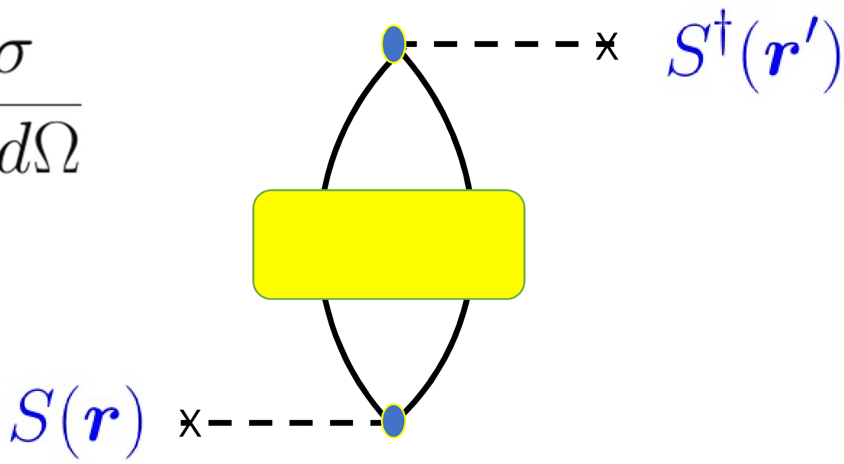
In the last line, Σ_X and Φ_X disappeared !

● Double differential cross section

When B is in continuum specified by X,

$$\frac{d^2\sigma}{d\omega^* d\Omega} = K\,\frac{\sqrt{s}}{m_A}$$
$$\times \int d^3\boldsymbol{r}' \int d^3\boldsymbol{r}\; S^\dagger(\boldsymbol{r}')\; R_\rho(\boldsymbol{r}', \boldsymbol{r}; \omega)\; S(\boldsymbol{r})$$

$$\frac{d^2\sigma}{d\omega^* d\Omega}$$



[Comment]

- PWBA vs. DWBA

PWBA

$$\frac{d^2\sigma}{d\omega^* d\Omega} = K \frac{\sqrt{s}}{m_A} |\tilde{V}(\boldsymbol{q}^*)|^2 R_\rho(\boldsymbol{q}^*, \omega)$$

DWBA

$$\frac{d^2\sigma}{d\omega^* d\Omega} = K \frac{\sqrt{s}}{m_A} \\ \times \int d^3\boldsymbol{r}' \int d^3\boldsymbol{r} \; S^*(\boldsymbol{r}') \; R_\rho(\boldsymbol{r}', \boldsymbol{r}; \omega) \; S(\boldsymbol{r})$$

Fourier Transform of $R_\rho(\boldsymbol{r}', \boldsymbol{r}; \omega)$

$$\tilde{R}_\rho(\boldsymbol{q}', \boldsymbol{q}; \omega) = \int d^3\boldsymbol{r}' \int d^3\boldsymbol{r} \mathrm{e}^{\mathrm{i}\boldsymbol{q}'\cdot\boldsymbol{r}'} R_\rho(\boldsymbol{r}', \boldsymbol{r}; \omega) \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}$$

Response function in PWBA

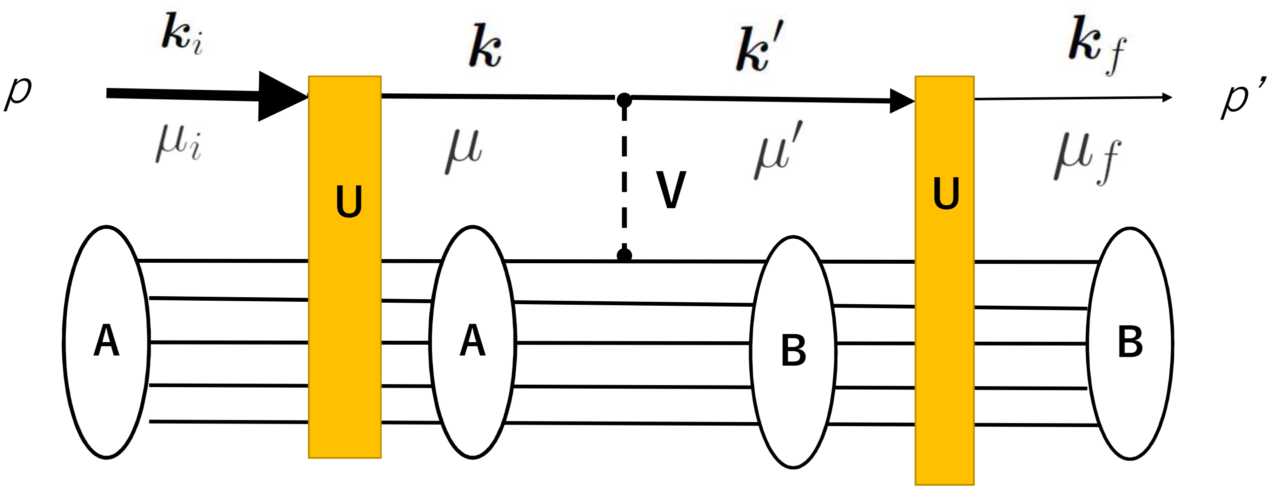
$$R_\rho(\boldsymbol{q}^*; \omega) = \tilde{R}_\rho(\boldsymbol{q}^*, \boldsymbol{q}^*; \omega)$$

2. (N, N') Reaction

Let's consider (N, N') reaction, where the spins are involved.

More concretely let's consider

$$A(p, p')B$$



At the transition, spin change is $\mu \longrightarrow \mu'$, not the observed change $\mu_i \longrightarrow \mu_f$

Generalization for cases with spins

● Optical potential

The optical potential has the spin-orbit force

$$U = U_0(r) + U_{ls}(r)\boldsymbol{\sigma}\cdot\boldsymbol{\ell}$$

● Distorted waves

$$\begin{aligned}\chi^{(+)}_{\boldsymbol{k}_i}(\boldsymbol{r}) \to \chi_{\boldsymbol{k}_i,\mu_i}(\boldsymbol{r}) &= \sum_{\mu}\chi^{(+)}_{\mu,\mu_i}(\boldsymbol{k}_i;\boldsymbol{r})|\mu\rangle \\ &= \begin{pmatrix}\chi^{(+)}_{+,\mu_i}(\boldsymbol{k}_i;\boldsymbol{r}) \\ \chi^{(+)}_{-,\mu_i}(\boldsymbol{k}_i;\boldsymbol{r})\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\chi^{(-)}_{\boldsymbol{k}_f}(\boldsymbol{r}) \to \chi_{\boldsymbol{k}_f,\mu_f}(\boldsymbol{r}) &= \sum_{\mu'}\chi^{(-)}_{\mu',\mu_f}(\boldsymbol{k}_f;\boldsymbol{r})|\mu'\rangle \\ &= \begin{pmatrix}\chi^{(-)}_{+,\mu_f}(\boldsymbol{k}_f;\boldsymbol{r}) \\ \chi^{(-)}_{-,\mu_f}(\boldsymbol{k}_f;\boldsymbol{r})\end{pmatrix}\end{aligned}$$

● Interaction

$$
\begin{aligned}
V &= \sum_k \{V_0(\boldsymbol{r}_0 - \boldsymbol{r}_k) + \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_k V_\sigma(\boldsymbol{r}_0 - \boldsymbol{r}_k)\} \\
&= \int d^3\boldsymbol{r}\ \ \{V_0(\boldsymbol{r}_0 - \boldsymbol{r})\rho(\boldsymbol{r}) \\
&\qquad\qquad + V_\sigma(\boldsymbol{r}_0 - \boldsymbol{r}) \sum_a \sigma_{a,0}\ \rho_a^\sigma(\boldsymbol{r})\Big\}
\end{aligned}
$$

with **Spin Density** operator

$$
\rho_a^\sigma(\boldsymbol{r}) = \sum_k \sigma_{a,k}\ \delta(\boldsymbol{r} - \boldsymbol{r}_k)
$$

● Outer Impulse Function

$S(\boldsymbol{r})$ is generalized as

$$S_{\mu'\mu_f,\mu\mu_i}(\boldsymbol{r}) = S^{(0)}_{\mu'\mu_f,\mu\mu_i}(\boldsymbol{r}) + S^{(\sigma)}_{a,\mu'\mu_f,\mu\mu_i}(\boldsymbol{r})$$

$$S^{(0)}_{\mu'\mu_f,\mu\mu_i}(\boldsymbol{r}) \\ = \delta_{\mu'\mu}\langle\chi^{(-)}_{\mu',\mu_f}(\boldsymbol{k}_f;\boldsymbol{r}_0)|V_0(\boldsymbol{r}_0-\boldsymbol{r})|\chi^{(+)}_{\mu,\mu_i}(\boldsymbol{k}_i;\boldsymbol{r}_0)\rangle$$

$$S^{(\sigma)}_{\mu'\mu_f,\mu\mu_i}(\boldsymbol{r}) \\ = [\sigma_a]_{\mu'\mu}\langle\chi^{(-)}_{\mu',\mu_f}(\boldsymbol{k}_f;\boldsymbol{r}_0)|V_\sigma(\boldsymbol{r}_0-\boldsymbol{r})|\chi^{(+)}_{\mu,\mu_i}(\boldsymbol{k}_i;\boldsymbol{r}_0)\rangle$$

● T-matrix

T-matrix also becomes a 2 x 2 matrix with respect to the incident and exit nucleon spin directions.

$$\begin{aligned} T_{fi} &= T_{\mu_f \mu_i}(\boldsymbol{k}_f, \boldsymbol{k}_i; B, A) \\ &= \int d^3\boldsymbol{r} \; \Big\{ S^{(0)}_{\mu_f \mu_i}(\boldsymbol{r}) \; F^{(0)}_{BA}(\boldsymbol{r}) \\ &\qquad\qquad + \sum_{a=x,y,z} S^{(\sigma)}_{a,\mu_f \mu_i}(\boldsymbol{r}) \; F^{(\sigma)}_{a,BA}(\boldsymbol{r}) \Big\} \end{aligned}$$

with the form factors

$$\begin{aligned} F^{(0)}_{BA}(\boldsymbol{r}) &= \langle B | \rho(\boldsymbol{r}) | A \rangle \\ F^{(\sigma)}_{a,BA}(\boldsymbol{r}) &= \langle B | \rho^{\sigma}_{a}(\boldsymbol{r}) | A \rangle \end{aligned}$$

● Differential Cross Section

The previous form

$$\frac{d\sigma}{d\Omega} = K \int d^3\boldsymbol{r}' \int d^3\boldsymbol{r} \\ \times \; S^*(\boldsymbol{r}') \left[F^*_{BA}(\boldsymbol{r}') F_{BA}(\boldsymbol{r}) \right] S(\boldsymbol{r})$$

is generalized as

$$\frac{d\sigma}{d\Omega} = K \frac{1}{2} \sum_{\mu_f,\mu_i} \int d^3\boldsymbol{r}' \int d^3\boldsymbol{r}$$
$$\times \left\{ S^{(0)}_{\mu_f\mu_i}(\boldsymbol{r}') \; F^{(0)}_{BA}(\boldsymbol{r}') + \sum_a S^{(\sigma)}_{a,\mu_f\mu_i}(\boldsymbol{r}') \; F^{(\sigma)}_{a,BA}(\boldsymbol{r}') \right\}^\dagger$$
$$\times \left\{ S^{(0)}_{\mu_f\mu_i}(\boldsymbol{r}) \; F^{(0)}_{BA}(\boldsymbol{r}) + \sum_a S^{(\sigma)}_{a,\mu_f\mu_i}(\boldsymbol{r}) \; F^{(\sigma)}_{a,BA}(\boldsymbol{r}) \right\}$$

In contrast to PWBA

$S^{(0)}_{\mu_f\mu_i}$ is no more proportional to $\delta_{\mu_f\mu_i}$

$S^{(\sigma)}_{a,\mu_f\mu_i}$ is no more proportional to $[\sigma_a]_{\mu_f\mu_i}$

Interference terms no more vanish.

● Response Functions

We generalize the notations of the spin space operators as

$$\sigma_0^{(0)} = 1, \qquad \sigma_a^{(1)} = \sigma_a, \quad (a = x, y, z)$$

and write the density operators as

$$\rho_a^{(\alpha)}(\boldsymbol{r}) = \sum_k \sigma_{a,k}^{(\alpha)} \delta(\boldsymbol{r} - \boldsymbol{r}_k)$$
$$(\alpha = 0, 1, \quad a = 0, x, y, z)$$

Thus the response functions are unifiedly expressed as

$$R_{a,b}^{\alpha,\beta}(\boldsymbol{r}', \boldsymbol{r}; \omega)$$
$$= -\frac{1}{\pi} \mathrm{Im} \langle \Phi_A | \rho_a^{(\alpha),\dagger}(\boldsymbol{r}') \frac{1}{\omega - H_A + \mathrm{i}\delta} \rho_b^{(\beta)}(\boldsymbol{r}) | \Phi_A \rangle$$

$$(\alpha, \beta = 0, 1, \quad a, b = 0, x, y, z)$$

● Inclusive double differential cross section

Now the previous form

$$\frac{d^2\sigma}{d\omega^* d\Omega} = K\,\frac{\sqrt{s}}{m_A} \int d^3\boldsymbol{r}' \int d^3\boldsymbol{r} \\ \times S^*(\boldsymbol{r}')\; R_\rho(\boldsymbol{r}', \boldsymbol{r}; \omega)\; S(\boldsymbol{r})$$

is generalized as

$$\frac{d^2\sigma}{d\omega^* d\Omega} = K\,\frac{\sqrt{s}}{m_A}\,\frac{1}{2} \sum_{\mu_f,\mu_i} \sum_{\alpha,\beta} \sum_{a,b} \int d^3\boldsymbol{r}' \int d^3\boldsymbol{r} \\ \times S^{(\alpha),\dagger}_{a,\mu_f\mu_i}(\boldsymbol{r}')\; R^{\alpha,\beta}_{a,b}(\boldsymbol{r}', \boldsymbol{r}; \omega)\; S^{(\beta)}_{b,\mu_f\mu_i}(\boldsymbol{r})$$

More concisely

$$\frac{d^2\sigma}{d\omega^* d\Omega} = \frac{K}{2}\,\frac{\sqrt{s}}{m_A} \sum_{\alpha,\beta} \sum_{a,b} \int d^3\boldsymbol{r}' \int d^3\boldsymbol{r} \\ \times \mathrm{Tr}\left[S^{(\alpha),\dagger}_{a}(\boldsymbol{r}')\; R^{\alpha,\beta}_{a,b}(\boldsymbol{r}', \boldsymbol{r}; \omega)\; S^{(\beta)}_{b}(\boldsymbol{r})\right]$$

● Angular momentum representation

Assume the spin of the target A is $J_A = 0$ (even-even nucleus), we can write

$$\frac{d^2\sigma}{d\omega^* d\Omega} = \frac{K}{2}\frac{\sqrt{s}}{m_A}\sum_J \sum_{S'L',S,L} \int r'^2 dr' \int r^2 dr \\ \times \mathrm{Tr}\left[S^\dagger_{JS'L'}(r')\; R^J_{S'L',SL}(r',r;\omega)\; S_{JSL}(r)\right]$$

J : Transferred total angular momentum

S : Transferred intrinsic spin

L : Transferred orbital angular momentum

See references:

K. Kawahigashi et al., PR **C63**, 044609(2001)

M. Ichimura, *Formalism for CRDW*,

http://www.nishina.riken.jp/researcher/archive/program_e.html

[Comments]

(1) Reaction and structure parts are
no more separated,
but the structure is still very simple.

(2) This simplicity is because the interaction
is of the two body and local.

(3) Interference between spin dependent and
independent forces remains.

VI. DWIA

Distorted Wave Impulse Approximation

To extract reliable information about nuclear structure, we must have reliable knowledge about the nuclear reaction part.

One important point is to adopt a realistic interaction.
Till now we did not specify the interaction V.

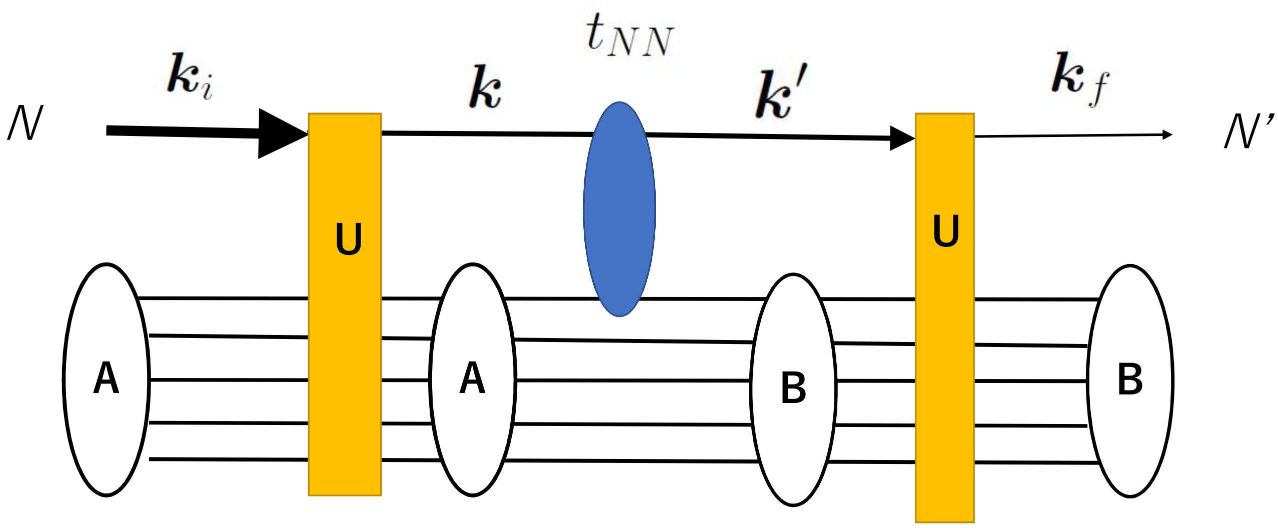
What interaction should we use ?

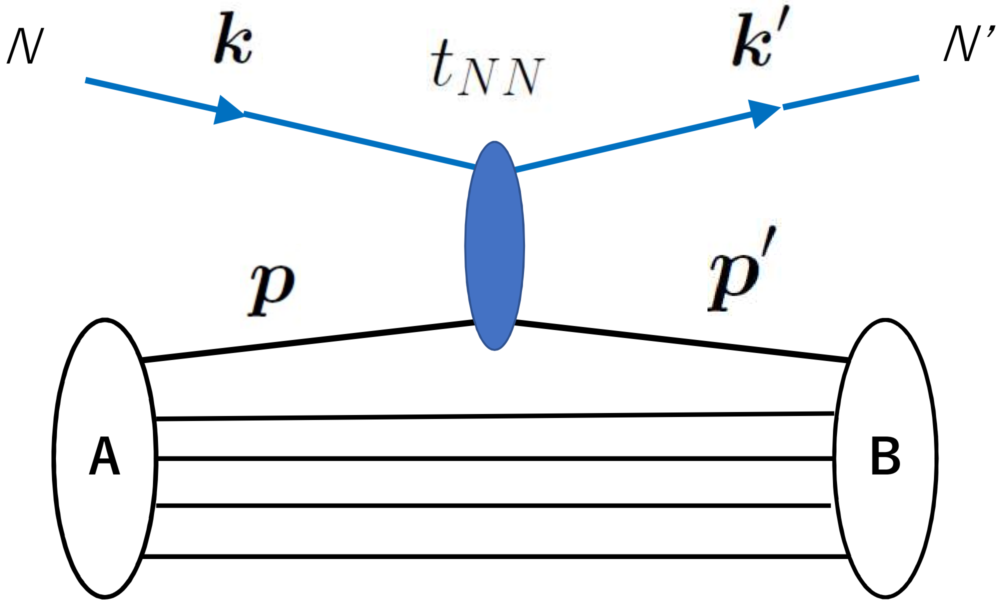
1. Impulse Approximation

An approximation often used
for high energy NA reactions is to use

the free NN t-matrix $t^{(0)}_{NN}$.
because its on-energy shell part is observed experimentally.

This is called **Impulse Approximation**





What we need is the matrix elements of t_{NN} in the NA cm frame

$$\langle \boldsymbol{k}', \boldsymbol{p}' | t_{NN}^{(0)} | \boldsymbol{k}, \boldsymbol{p} \rangle_{NA}$$

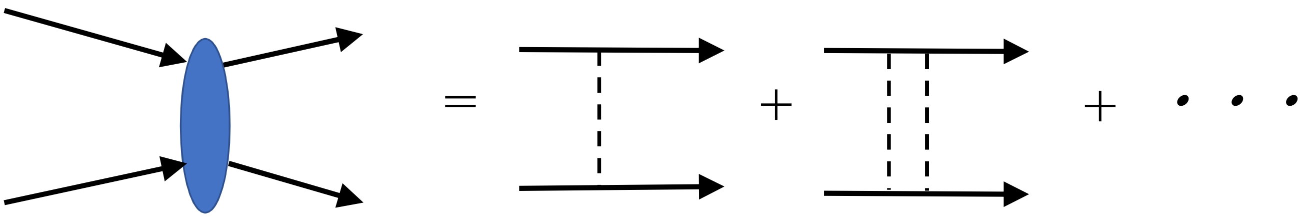
Using the momentum conservation, we write

$$\langle \boldsymbol{k}', \boldsymbol{p}' | t_{NN}^{(0)} | \boldsymbol{k}, \boldsymbol{p} \rangle_{NA} = t_{NN}(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{k})$$

where

$$\boldsymbol{q} = \boldsymbol{k}' - \boldsymbol{k}, \qquad \boldsymbol{p}' = \boldsymbol{p} - \boldsymbol{q}$$

● What is the difference between t_{NN} and V in the previous sections ?



○ Crucial difference

- V is a local operator (2-point function)
- t_{NN} is a non-local operator (4-point function)

In the momentum representation

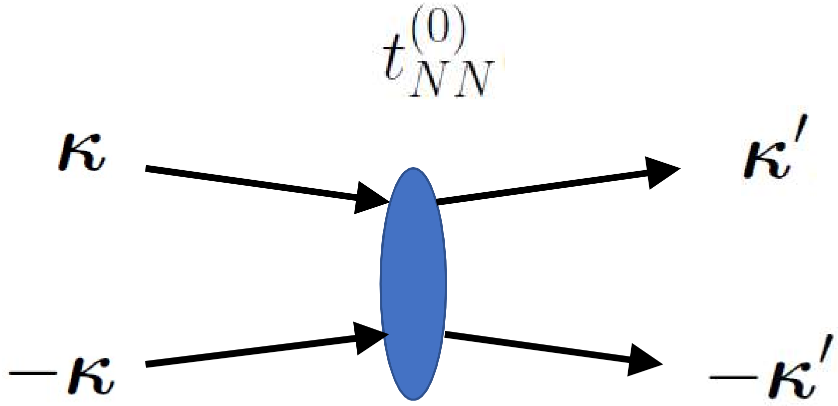
- $V = \tilde{V}(\boldsymbol{q})$: depends on only $\boldsymbol{q}$
- $t_{NN} = t_{NN}(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{k})$: on 3 momenta.

To use the DWBA formulas,
we have to make t_{NN} local.
We will discuss later.

2. Free NN t-matrix

The free NN scattering t-matrix in the NN cm frame is written as

$$\langle \boldsymbol{\kappa}', -\boldsymbol{\kappa}' | t_{NN}^{(0)} | \boldsymbol{\kappa}, -\boldsymbol{\kappa} \rangle_{NN} = t_{NN}^{(0)}(\boldsymbol{\kappa}', \boldsymbol{\kappa})$$



NN scattering data give us only the on-energy shell components of $t_{NN}^{(0)}(\boldsymbol{\kappa}', \boldsymbol{\kappa})$ namely, at

$$\kappa = \kappa'$$

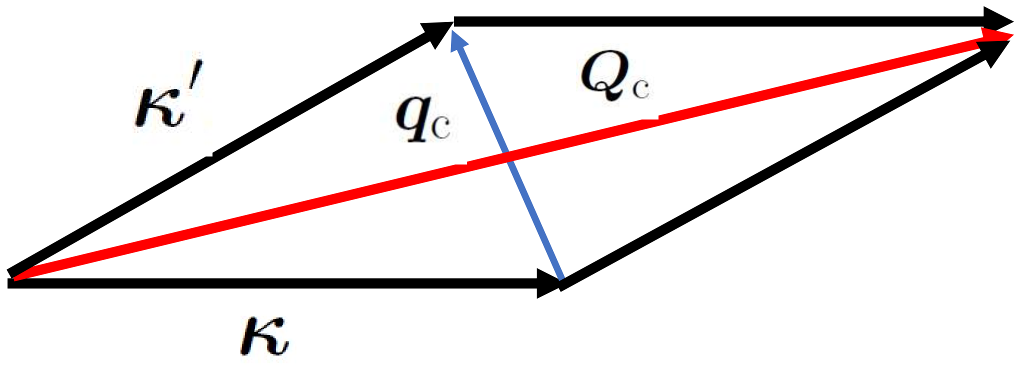
(Neglected proton-neutron mass difference.)

● General form of t_{NN}

There are various expressions for t_{NN}.
A typical one we use is the KMT expression.

Introduce the three vectors

$$\boldsymbol{q}_{\rm c} = \boldsymbol{\kappa}' - \boldsymbol{\kappa},$$
$$\boldsymbol{Q}_{\rm c} = \boldsymbol{\kappa}' + \boldsymbol{\kappa},$$
$$\hat{\boldsymbol{n}}_{\rm c} = \frac{\boldsymbol{\kappa} \times \boldsymbol{\kappa}'}{|\,\boldsymbol{\kappa} \times \boldsymbol{\kappa}'\,|}$$



KMT write $t_{NN}^{(0)}$ as

$$\begin{aligned} t_{k,\mathrm{NN}}^{(0)}(\boldsymbol{\kappa}', \boldsymbol{\kappa}) = A & \\ & + B(\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{n}}_{\mathrm{c}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{n}}_{\mathrm{c}}) \\ & + C\left\{(\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{n}}_{\mathrm{c}}) + (\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{n}}_{\mathrm{c}})\right\} \\ & + E(\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{q}}_{\mathrm{c}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{q}}_{\mathrm{c}}) \\ & + F(\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{Q}}_{\mathrm{c}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{Q}}_{\mathrm{c}}) \end{aligned}$$

where

$$A = A_0 + A_1 \boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k,$$

$$B = B_0 + B_1 \boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k,$$

$$\text{etc.}$$

The amplitudes, $A_i, B_i, \cdots$ are scalar functions of 3-independent scalars, e.g.

$$A_i = A_i(q_c^2,\ Q_c^2,\ \boldsymbol{q}_c \cdot \boldsymbol{Q}_c), \quad \cdots$$

A.K.Kerman, H.McManus and R.M.Thaler, Ann. of Phys. **8** (1959) 551

[Just for fun]

Full KMT expression

$$
\begin{aligned}
& t^{(0)}_{k,\mathrm{NN}}(\boldsymbol{\kappa}', \boldsymbol{\kappa}) \\
= & A \\
+ & B(\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{n}}_{\mathrm{c}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{n}}_{\mathrm{c}}) \\
+ & C\left\{(\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{n}}_{\mathrm{c}}) + (\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{n}}_{\mathrm{c}})\right\} \\
+ & D\left\{(\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{q}}_{\mathrm{c}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{Q}}_{\mathrm{c}}) + (\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{Q}}_{\mathrm{c}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{q}}_{\mathrm{c}})\right\} \\
+ & E(\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{q}}_{\mathrm{c}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{q}}_{\mathrm{c}}) \\
+ & F(\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{Q}}_{\mathrm{c}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{Q}}_{\mathrm{c}})
\end{aligned}
$$

D-term vanishes on the energy shell, and neglect throughout this lecture.

3. Off-energy-shell Extrapolation

Experimental data give us only the on-energy-shell amplitudes at the incident energy $T_i^{\rm lab}$ in the laboratory frame

$$A_i = A_i(q_c^2,\ Q_c^2,\ \boldsymbol{q}_c \cdot \boldsymbol{Q}_c = 0) = A_i(\theta_{NN}, T_i^{\rm lab})$$

$$\cdots\cdots$$

because

$$\boldsymbol{q}_c \cdot \boldsymbol{Q}_c = \kappa'^2 - \kappa^2 = 0$$

on the energy-shell.

To calculate t_{NN} in the NA scattering, we need data at off the energy-shell. where

$$E_N(k) + E_N(p) \neq E_N(k') + E_N(p')$$

in general.

How to get them ?

● Love-Franey Prescription

There are several methods.
Here I only explain the Love-Franey method.

They introduced a trial potential for the given incident energy $T_i^{\rm lab}$

$$V^{\rm LF}(T_i^{\rm lab}) = V_{\rm SO}^{\rm C} + V_{\rm SE}^{\rm C} + V_{\rm TO}^{\rm C} + V_{\rm TE}^{\rm C} + V^{\rm LSO} + V^{\rm LSE} + V^{\rm TNO} + V^{\rm TNE}$$

SO : singlet odd, SE : singlet even
TO : triplet odd TE : triplet even
LSE : even LS , LSO : odd LS
TNO : odd tensor, TNE : even tensor

Each has several adjustable parameters.

W.G. Love and M.A. Franey, Phys. Rev. **C24**, 1073 (1981),
M.A. Franey and W.G. Love, Phys. Rev. **C31**, 488 (1985),

Calculate the on-energy-shell NN t-matrix at the given $T_i^{\rm lab}$ by Born approximation.

$$t_{NN}^{\rm LF}(\boldsymbol{q}_c, \boldsymbol{Q}_c, T_i^{\rm lab}) = \langle \boldsymbol{\kappa}' | V^{\rm LF} | \boldsymbol{\kappa} \rangle_D \pm \langle \boldsymbol{\kappa}' | V^{\rm LF} | \boldsymbol{\kappa} \rangle_E$$

Adjust the parameters of $V^{\rm LF}$ to reproduce the observed on-shell $t_{NN}^{(0)}$

$$t_{NN}^{(0)}(\boldsymbol{\kappa}', \boldsymbol{\kappa}) = t_{NN}^{\rm LF}(\boldsymbol{q}_c, \boldsymbol{Q}_c, T_i^{\rm lab})$$

Using the obtained potential, we can calculate the amplitudes

$$A_i = A_i(q_c^2,\ Q_c^2,\ \boldsymbol{q}_c \cdot \boldsymbol{Q}_c), \quad \cdots$$

even off the energy-shell.

[Just for fun]

$$
\begin{aligned}
& t^{\mathrm{LF}}_{k,\mathrm{NN}}(\boldsymbol{q}_c, \boldsymbol{Q}_c) \\
&= \left[\tilde{V}^{\mathrm{C}}_{\mathrm{SO}}(q_{\mathrm{c}}) - \tilde{V}^{\mathrm{C}}_{\mathrm{SO}}(Q_{\mathrm{c}})\right] P_{S=0} P_{T=0} \\
&+ \left[\tilde{V}^{\mathrm{C}}_{\mathrm{SE}}(q_{\mathrm{c}}) + \tilde{V}^{\mathrm{C}}_{\mathrm{SE}}(Q_{\mathrm{c}})\right] P_{S=0} P_{T=1} \\
&+ \left[\tilde{V}^{\mathrm{C}}_{\mathrm{TO}}(q_{\mathrm{c}}) - \tilde{V}^{\mathrm{C}}_{\mathrm{TO}}(Q_{\mathrm{c}})\right] P_{S=1} P_{T=1} \\
&+ \left[\tilde{V}^{\mathrm{C}}_{\mathrm{TE}}(q_{\mathrm{c}}) + \tilde{V}^{\mathrm{C}}_{\mathrm{TE}}(Q_{\mathrm{c}})\right] P_{S=1} P_{T=0} \\
&+ \frac{\mathrm{i}}{4}\left[Q_{\mathrm{c}} \tilde{V}^{\mathrm{LSO}}(q_{\mathrm{c}}) + q_{\mathrm{c}} \tilde{V}^{\mathrm{LSO}}(Q_{\mathrm{c}})\right] \left((\boldsymbol{\sigma}_0 + \boldsymbol{\sigma}_k) \cdot \hat{\boldsymbol{n}}_{\mathrm{c}}\right) P_{T=1} \\
&+ \frac{\mathrm{i}}{4}\left[Q_{\mathrm{c}} \tilde{V}^{\mathrm{LSE}}(q_{\mathrm{c}}) - q_{\mathrm{c}} \tilde{V}^{\mathrm{LSE}}(Q_{\mathrm{c}})\right] \left((\boldsymbol{\sigma}_0 + \boldsymbol{\sigma}_k) \cdot \hat{\boldsymbol{n}}_{\mathrm{c}}\right) P_{T=0} \\
&- \left[\tilde{V}^{\mathrm{TNO}}(q_{\mathrm{c}}) S_{0k}(\hat{\boldsymbol{q}}_{\mathrm{c}}) - \tilde{V}^{\mathrm{TNO}}(Q_{\mathrm{c}}) S_{0k}(\hat{\boldsymbol{Q}}_{\mathrm{c}})\right] P_{T=1} \\
&- \left[\tilde{V}^{\mathrm{TNE}}(q_{\mathrm{c}}) S_{0k}(\hat{\boldsymbol{q}}_{\mathrm{c}}) + \tilde{V}^{\mathrm{TNE}}(Q_{\mathrm{c}}) S_{0k}(\hat{\boldsymbol{Q}}_{\mathrm{c}})\right] P_{T=0}
\end{aligned}
$$

where

$$
P_{S=0} = \frac{1 - \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_k}{4}, \qquad P_{S=1} = \frac{3 + \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_k}{4}
$$

and similarly for $P_{T=0}$ and $P_{T=1}$.

$$
S_{0k}(\hat{\boldsymbol{q}}) = 3(\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{q}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{q}}) - (\boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_k)
$$

4. Frame Transformation

To obtain the t_{NN} in the NA cm frame, we have to transform the $t_{NN}^{(0)}$ in the NN cm frame to that in the NA cm frame.

Lorentz transformation gives

$$t_{NN}(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{k}) = \langle \boldsymbol{k'}, \boldsymbol{p'} | t_{NN}^{(0)} | \boldsymbol{k}, \boldsymbol{p} \rangle_{NA}$$

$$= \sqrt{\frac{E_N(\kappa)E_N(\kappa)E_N(\kappa')E_N(\kappa')}{E_N(k)E_N(p)E_N(k')E_N(p')}}$$

$$\times R_{sp}^{l}(\boldsymbol{k'}, \boldsymbol{p'}) \langle \boldsymbol{\kappa'}, -\boldsymbol{\kappa'} | t_{NN}^{(0)} | \boldsymbol{\kappa}, -\boldsymbol{\kappa} \rangle_{NN} R_{sp}^{r}(\boldsymbol{k}, \boldsymbol{p})$$

where

$$E_N(p) = \sqrt{m_N^2 + p^2}$$

and $R_{sp}^{l}(\boldsymbol{k'}, \boldsymbol{p'})$, $R_{sp}^{r}(\boldsymbol{k}, \boldsymbol{p})$ are the relativistic spin rotation matrices.

Neglecting the relativistic spin rotation

$$R^l_{sp}(\boldsymbol{k}', \boldsymbol{p}') = 1, \quad R^r_{sp}(\boldsymbol{k}, \boldsymbol{p}) = 1$$

we get

$$t_{NN}(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{k}) = J(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{k})\; t^{(0)}_{NN}(\boldsymbol{\kappa}', \boldsymbol{\kappa})$$

with the Möller factor

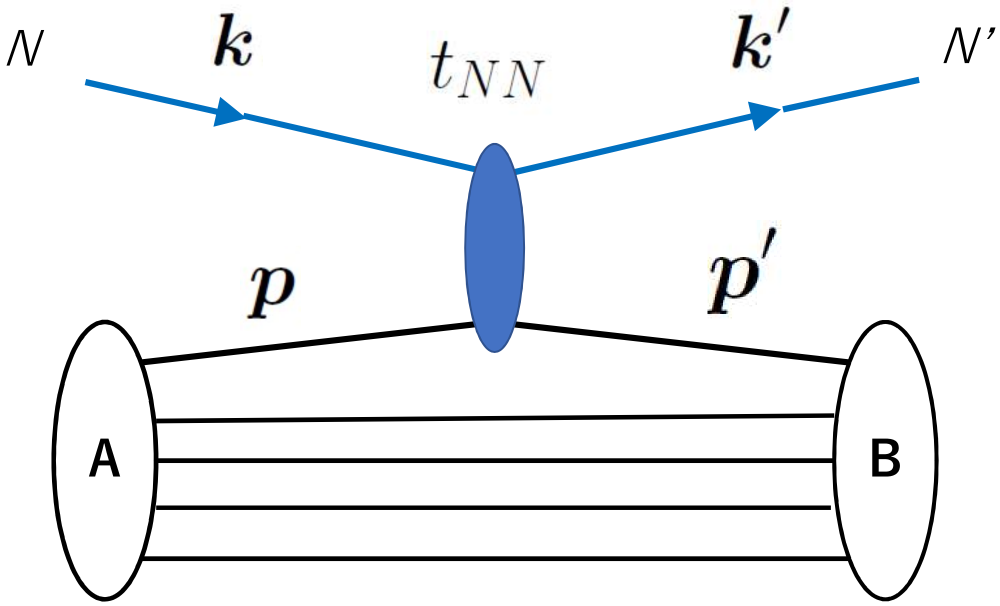
$$J(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{k}) = \frac{E_N(\kappa)E_N(\kappa')}{\sqrt{E_N(k)E_N(p)E_N(k')E_N(p')}}$$

5. Local Potential Approximation

Now we get t_{NN} in the NA cm frame as

$$t_{k,NN}(\boldsymbol{q}, \boldsymbol{p}_k, \boldsymbol{k}) = J(\boldsymbol{q}, \boldsymbol{p}_k, \boldsymbol{k})\, t_{NN}^{\rm LF}(\boldsymbol{q}_c, \boldsymbol{Q}_c, T_i^{\rm lab})$$

$(\boldsymbol{q}_c, \boldsymbol{Q}_c, T_i^{\rm lab})$ are determined by $(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{k})$



To use the response function method presented in the previous sections, $t_{k,NN}(\boldsymbol{q}, \boldsymbol{p}_k, \boldsymbol{k})$ should be approximated to be a local operator !

Following approximations are often used.

5.1 Representative Momentum Approximation

In the calculation of

$$\langle \Phi_B | \sum_k t_{k,NN}(\boldsymbol{q}, \boldsymbol{p}_k, \boldsymbol{k}) | \Phi_A \rangle$$

Integration of $\boldsymbol{p}_k$ is very cumbersome.

● Replace $\boldsymbol{p}_k$ by the suitably chosen representative momentum $\bar{\boldsymbol{p}}$

$$\boldsymbol{p}_k \;\longrightarrow\; \bar{\boldsymbol{p}}$$

Love-Franey's choice

$$\bar{\boldsymbol{p}} = -\frac{\boldsymbol{k}_i}{A}$$

5.2 Asymptotic Momentum Approx.

In the calculation of

$$\langle \chi_{\boldsymbol{k}_f} | \sum_k t_{k,NN}(\boldsymbol{q}, \bar{\boldsymbol{p}}, \boldsymbol{k}) | \chi_{\boldsymbol{k}_i} \rangle$$

Integration of $\boldsymbol{k}$ is troublesome.

- Replace $\boldsymbol{k}$ by its asymptotic value $\boldsymbol{k}_i$

$$\boldsymbol{k} \longrightarrow \boldsymbol{k}_i$$

for those which weekly depend on k, such as Möller factor and the amplitude $A_i, B_i, \cdots$.

5.3. Pseudo-potential Approx.

Now we reached

$$t_{k,NN}(\boldsymbol{q}, \boldsymbol{p}_k, \boldsymbol{k}) \approx J_{\rm LF} t^{\rm LF}_{k,{\rm NN}}(\boldsymbol{q}_c, \boldsymbol{Q}_c, T^{\rm lab}_i)$$

This is still non-local due to the presence of the exchange terms.

- Apply the pseudo-potential approx. to the exchange terms. Namely replace

$$Q_c \;\longrightarrow\; k_i$$

for the amplitudes $A_i, B_i, \cdots$

- As to the direction, we assume $\hat{\boldsymbol{Q}}_c$ is approximately perpendicular to $\hat{\boldsymbol{q}}_c$ as on the energy shell.

$$\hat{\boldsymbol{Q}}_c \perp \hat{\boldsymbol{q}}_c$$

5.4 Additional approximations

● Forward scattering approximation for the the Möller factor.

Love-Franey evaluated the Möller factor at $\boldsymbol{q} = 0$, namely, forward elastic scattering.

$$
\begin{aligned}
J(\boldsymbol{q}, \boldsymbol{p}_k, \boldsymbol{k}) &\approx J(\boldsymbol{q} = 0, \boldsymbol{p}_k = -\frac{\boldsymbol{k}_i}{A}, \boldsymbol{k} = \boldsymbol{k}_i) \\
&= J_{\mathrm{LF}} = \frac{s_{NN}}{4E_N(k_i)E_N(k_i/A)}
\end{aligned}
$$

with

$$
s_{NN} = (E_N(k_i) + E_N(k_i/A))^2 - \left(\frac{A-1}{A}\right)^2 k_i^2
$$

Now the Möller factor becomes a constant for the given incident energy

● Non-relativistic approximation for the momentum transfer

$$\boldsymbol{q}_c = \boldsymbol{\kappa}' - \boldsymbol{\kappa} \approx \boldsymbol{k}' - \boldsymbol{k} = \boldsymbol{q}$$

5.5 Local potential approximation

Finally we get t_{NN} in the NA cm frame in a local form as

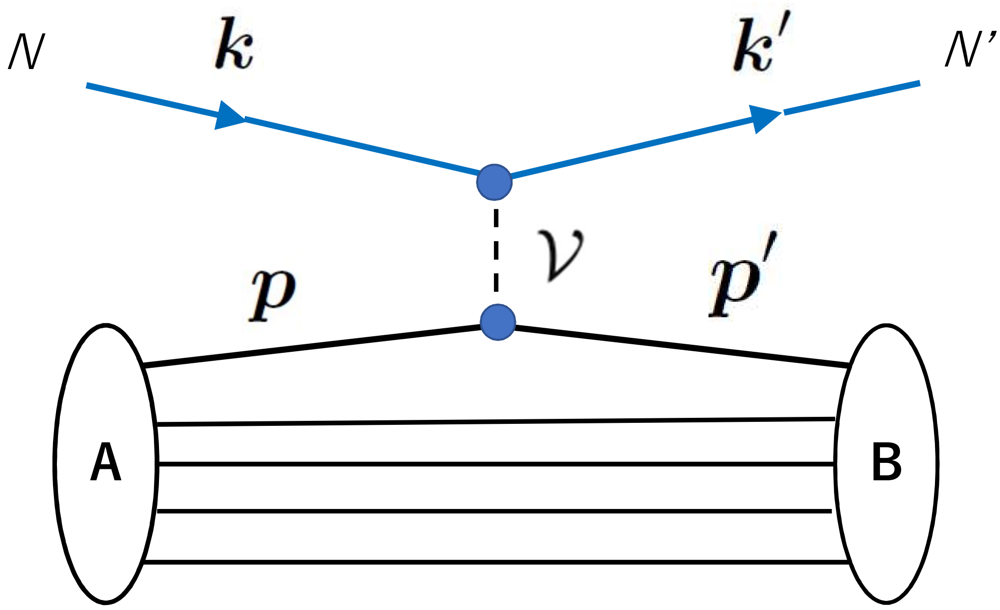
$$\begin{aligned} t_{k,NN}(\boldsymbol{q}, \boldsymbol{p}_k, \boldsymbol{k}) &\approx J_{\mathrm{LF}}\ t^{\mathrm{LF}}_{k,NN}(\boldsymbol{q}, \boldsymbol{k}_i) \\ &= J_{\mathrm{LF}}\ t^{\mathrm{LF}}_{k,NN}(\boldsymbol{q}, T_i^{\mathrm{lab}}) \end{aligned}$$

This is the function only of $\boldsymbol{q}$ for the given incident energy.

Moving to the coordinate space,
t_{NN} is effectively calculated by the direct term of the energy dependent local potential

$$\mathcal{V} = \sum_k V^{\mathrm{LF}}(\boldsymbol{r}_0 - \boldsymbol{r}_k; k_i) + V_{\mathrm{pseudo}}$$

Use this potential for DWIA (PWIA), and calculate only the direct term.



6. PWIA

Plane Wave Impulse Approximation

Rewrite $J_{\rm LF}\, t^{\rm LF}_{k,NN}(\boldsymbol{q}, T_i^{\rm lab})$ in the form of

$$
\begin{aligned}
t_{k,\mathrm{NN}}(\boldsymbol{q}, \boldsymbol{k}_i) = {} & A'(q, T_i^{\rm lab}) \\
& + B'(q, T_i^{\rm lab})(\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{n}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{n}}) \\
& + C'(q, T_i^{\rm lab})\,\{(\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{n}}) + (\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{n}})\} \\
& + E'(q, T_i^{\rm lab})(\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{q}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{q}}) \\
& + F'(q, T_i^{\rm lab})(\boldsymbol{\sigma}_0 \cdot \hat{\boldsymbol{p}})(\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{p}})
\end{aligned}
$$

Note in PWIA
$\boldsymbol{k}, \boldsymbol{q}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{p}}$ are all given by $\boldsymbol{k}_i, \boldsymbol{k}_f$.

M. Ichimura and K. Kawahigashi, PR **C45**, 1822(1992)

Recall the PWBA formula

$$
\begin{aligned}
T_{fi} = & \tilde{V}_{\tau}(\boldsymbol{q}^*)\ \delta_{\mu_f,\mu_i} F_{BA}^{(-)}(\boldsymbol{q}^*) \\
& + \tilde{V}_{\tau\sigma}(\boldsymbol{q}^*)\ \sum_a \left[\sigma_a\right]_{\mu_f,\mu_i} F_{BA}^{(-,a)}(\boldsymbol{q}^*)
\end{aligned}
$$

We get T-matrix in PWIA as

$$
T_{fi} = T_0 + T_n\sigma_n + T_q\sigma_q + T_p\sigma_p
$$

$$
\begin{aligned}
T_0 = & \Big(A'(q, T_i^{\mathrm{lab}}) + C'(q, T_i^{\mathrm{lab}})\Big) \\
& \times \langle\Phi_B|\sum_k \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k}|\Phi_A\rangle \\
T_n = & \Big(B'(q, T_i^{\mathrm{lab}}) + C'(q, T_i^{\mathrm{lab}})\Big) \\
& \times \langle\Phi_B|\sum_k (\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{n}})\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k}|\Phi_A\rangle \\
T_q = & E'(q, T_i^{\mathrm{lab}}) \\
& \times \langle\Phi_B|\sum_k (\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{q}})\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k}|\Phi_A\rangle \\
T_p = & F'(q, T_i^{\mathrm{lab}}) \\
& \times \langle\Phi_B|\sum_k (\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{p}})\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k}|\Phi_A\rangle
\end{aligned}
$$

Recall the cross section formula in PWBA

We get

$$\frac{d^2\sigma}{d\omega^* d\Omega} = ID_0 + ID_n + ID_q + ID_p$$

$$ID_0 = K\ \left|A'(q, T_i^{\mathrm{lab}}) + C'(q, T_i^{\mathrm{lab}})\right|^2 R_0(q,\omega)$$
$$ID_n = K\ \left|B'(q, T_i^{\mathrm{lab}}) + C'(q, T_i^{\mathrm{lab}})\right|^2 R_n(q,\omega)$$
$$ID_q = K\ \left|E'(q, T_i^{\mathrm{lab}})\right|^2 R_q(q,\omega)$$
$$ID_p = K\ \left|F'(q, T_i^{\mathrm{lab}})\right|^2 R_p(q,\omega)$$

$$R_0(q,\omega) = \sum_X |\langle\Phi_X| \sum_k \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k}|\Phi_A\rangle|^2 \delta(\omega - E_x^X)$$
$$R_n(q,\omega) = \sum_X |\langle\Phi_X| \sum_k (\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{n}})\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k}|\Phi_A\rangle|^2 \delta(\omega - E_x^X)$$
$$R_q(q,\omega) = \sum_X |\langle\Phi_X| \sum_k (\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{q}})\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k}|\Phi_A\rangle|^2 \delta(\omega - E_x^X)$$
$$R_p(q,\omega) = \sum_X |\langle\Phi_X| \sum_k (\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{p}})\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k}|\Phi_A\rangle|^2 \delta(\omega - E_x^X)$$

Commonly used Response Functions

(1) **Spin-scalar Response Function**

$$R_{\mathrm{S}}(q,\omega) = \sum_X |\langle\Phi_X| \sum_k \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k} |\Phi_A\rangle|^2 \\ \times \delta(\omega - E_x^X)$$

(2) **Spin-longitudinal Response Fun.**

$$R_{\mathrm{L}}(q,\omega) = \sum_X |\langle\Phi_X| \sum_k (\boldsymbol{\sigma}_k \cdot \hat{\boldsymbol{q}}) \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k} |\Phi_A\rangle|^2 \\ \times \delta(\omega - E_x^X)$$

(3) **Spin-transverse Response Fun.**

$$R_{\mathrm{T}}(q,\omega) = \frac{1}{2} \sum_X |\langle\Phi_X| \sum_k [\boldsymbol{\sigma}_k \times \hat{\boldsymbol{q}}]\, \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k} |\Phi_A\rangle|^2 \\ \times \delta(\omega - E_x^X)$$

They are related with R_0, R_q, R_n, R_p as

$$R_0 = R_{\mathrm{S}}, \quad R_q = R_{\mathrm{L}}, \quad R_n = R_p = R_{\mathrm{T}}$$